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Analytic Innovation and Derivation in Financial Engineering of Interest Rate Intensity Modelling and Computations

Gbenga Michael Ogungbenle[1*], Wipuni Sirisena[2], and Chukwunenye Ukwu[2]

[1*]Department of Actuarial Science, Faculty of Management Sciences, University of Jos, Nigeria
[2]Department of Mathematics, Faculty of Natural Sciences, University of Jos, Nigeria
Gbenga Michael Ogungbenle[1*],(https://orcid.org/0000-0001-5700-7738; moyosiolorun@gmail.com);
Wipuni Sirisena[2], (swipuni@yahoo.com, 0009000801780068),
Chukwunenye Ukwu[2] (0009000699223774, ukwu@unijos.edu.ng)
Corresponding author: moyosiolorun@gmail.com

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Abstract

The force of interest is a tool used to describe the instantaneous rate of growth or decline of life insurance asset over time. The core problem of this paper centers on how the instantaneous interest rate is modelled employing bivariate Taylor's series expansion to reflect the influence of multiple interdependent factors and what implications does it have for the valuation of life insurance products. This paper explores the theoretical foundations, analytic behaviour, calibration methodology and the empirical performance of this novel model providing a more flexible and accurate approach to capturing the time-varying nature of interest rates. By extending the traditional one-dimensional Taylor series expansion, we derive a novel closed-form expression for the force of interest while incorporating the investment period horizon and allowing for the estimation of more complex, non-linear interest rate behaviors. This method analytically enables the modeling of interest rates that exhibits time-dependent changes. The two-dimensional expansion offers a powerful tool for computing the present value of future cash flows and assessing risk in financial portfolios which may be applicable in a variety of financial contexts such as bond pricing, loan amortization and pension fund valuation, highlighting its potential to enhance the robustness of traditional interest rate models

Keywords: Investment, Valuations, Deterministic, Taylor's Series, Bivariate Interest Function

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1.0 Introduction

Interest rate intensity defines the instantaneous continuously compounded interest rate and represents the limiting value of the interest rate over an infinitesimally small time-interval. It is not directly observable in the deterministic insurance market, nevertheless it is modelled and inferred. In actuarial finance, the interest rate intensity assumes a pivotal role in modelling expected present value of cash flows and valuing life insurance products. The classical interest rate intensity models commonly assume deterministic or single variable form usually restricted to functions of time or a single economic factor. Nevertheless, current financial environments are characterised by multidimensional and dynamically interacting factors such as inflation, market volatility and interest rate regimes. These factors necessitate more sophisticated models which can capture the joint influence of multiple variables on interest rate dynamics. Despite the vast actuarial literature on life insurance valuation, there remains a critical gap in the direct application of multivariate Taylor's series expansions to model the interest rate intensity as a function of two or more interacting variables. Taylor's series offers a good analytic tool to estimate the nonlinear behaviour and interdependencies within a local region, yet their potential in the context of interest rate modelling is unexplored. This paper is motivated to address this gap by deriving a new form of interest rate intensity employing Taylor's series in two variables. The method introduces local estimations which incorporates sensitivity to changes in underlying drivers thereby offering a deeper and flexible framework for actuarial valuations. The lack of existing literature of such models highlights a compelling opportunity to contribute fundamentally new theoretical advancements and practical tools for the actuarial community.

The consistency conditions in financial and actuarial modeling are essential to ensure that models are realistic, mathematically sound, and aligned with the real behavior of financial instruments or systems. Specifically, in the context of instantaneous interest $\delta(t)$ for all t these conditions guarantee that the model does not produce contradictions, unrealistic outcomes, or illogical results. According to Parmenter (1999) and Chan and Tse (2018), the consistency condition is typically applied to the force of interest under the following assumptions. (i) $\delta(t) \geq 0$ for all t. This is the non-negativity condition this condition requires that the force of interest (the instantaneous rate at which value grows) cannot be negative. A negative force of interest would imply that the value of money is decreasing continuously over time, which is not common in standard financial environments, except in specific deflationary scenarios.

Following the arguments of Jothi (2009) and Regis (2019), if $\delta(t)$ were negative, then the value of a sum invested would decay over time, which contradicts the idea that money typically grows due to positive interest rates. This condition helps maintain a model that reflects standard financial systems where investments usually appreciate over time, (ii) $\delta(t)$ should be a continuous function of time. This condition ensures that the force of interest does not have jumps or discontinuities. If the force of interest were discontinuous, the accumulated value of an investment would change abruptly, leading to unrealistic scenarios. Financial models typically assume smooth, gradual changes in interest rates or economic conditions over time. Abrupt changes would suggest a catastrophic or unpredictable market event, which would be difficult to model consistently over time (Brusov *et al.*, 2023). However, Udoye *et al.* (2021) argued that this monotonicity condition implies that the force of interest may not exhibit oscillations or erratic changes in direction over time. For example, a non-decreasing force of interest would mean that interest rates are either stable or increasing, which is a typical assumption in economic environments that expect growth over time. $\delta(t)$ is non-decreasing, it means that the force of interest increases or remains stable as time progresses, reflecting an economy where the returns on investment tend to increase. $\delta(t)$ is non-increasing, it suggests an environment where returns diminish over time (such as a recessionary economy). The consistency condition here ensures that the force of interest leads to a proper discounting framework. If the force of interest $\delta(t)$ is inconsistent with the discounting mechanism, it would lead to contradictory pricing or valuation. For instance, if the force of interest is negative or erratic, it could produce an improper or non-intuitive discounting pattern, which would undermine the model's usefulness in pricing future cash flows or liabilities, (vii) the force of interest must be consistent with the broader economic environment, including factors like inflation, risk-free rates, and general market conditions. In practice, the force of interest should align with observable economic conditions. It reflects the rate at which value is expected to grow or decline in a particular financial or economic context. If the force of interest assumes rates that are much higher or lower than actual market rates, the model may yield unrealistic projections. This condition ensures that the model remains economically plausible and closely mirrors the behavior of real market financial instruments.

Rather than conducting an approximate numerical experiment to test a reasonable level of estimate for interest rate intensity, extant literature solely relies on the relation $\delta = \ln(1+i)$. Furthermore, actuarial literature does not show any evidence of development of this area. Consequently, one loses track of the correct numerical estimates. This study adopts a holistic treatment of the consistent properties of the force of interest in relation to Bernoulli series. The major reason behind the choice of Bernoulli series is that under mild conditions, it is computationally efficient in the performance of valuations involving discounting the insured's benefits. Following Parmenter (1999), the theory of life contingencies has been deterministically developed in most actuarial literature on the assumption that mortality occurs in accordance with a priori defined mortality table and that interest rate is assumed constant. Nonetheless, the functional theory of life contingencies implicitly addresses issues plaguing the stochastic behaviour of mortality rates such that the interest rates applicable to those conservative assumptions are enforced. The basic step is to consider the time until death as a random variable while interest rate is assumed constant.

According to McCutcheon and Scott (1986) and Kellison (2009), rates of return computed in terms of continuous interest rates are expressed as nominal rate of return for the purpose of present value calculations. However, it was observed in Panjer and Bellhouse (1980), Belhouse and Panjer (1981) and Anggraeni *et al.* (2019), that changes in effective interest rates can markedly affect the expected value of insurance

liabilities which interpreted as a numerical estimate of the amount the life office would charge today (present value of future cash flows) so as to meet future obligations promised to the scheme holder under defined conditions. The impact of the changes could be intractable due to the fact that it may be difficult to estimate, since interest rate is apparently subject to variation within an interval of time due to the vagaries of the prevailing economic conditions. In Kellison (1991), interest rate analysis normally starts with the theory of accumulation function $A(s,t)$ that gives the accumulated value at times t, $t > s > 0$, of a unit of fund invested at time s and maturing at time t. The accumulation function under compound interest exhibits the exponential form
To address this issue of variability analytically, a two-dimensional accumulation function $A(s_1, s_2)$ describing the amount that one unit of life fund investment at time s_1 accumulates to at time s_2 will be used. Let us assume an interval of time $s_0 < s_1 < s_2$ under consistency conditions, for an insurance fund investment of 1 unit at the initial time s_0. Following Kellison (2009), the total return on investment at s_2 is given as $A(s_0, s_2)$ if investment is made for the horizon length $s_2 - s_0$ starting from time s_0 and terminating at time s_2. In other words the total return is $A(s_0, s_1)A(s_1, s_2)$ if one invests at time s_0 for a term $s_1 - s_0$ and then at time s_1 reinvests the proceeds for a term $s_2 - s_1$. Under the no arbitrage conditions, the insurance transactions will not depend on the course of action taken by the life office hence the mathematical formulation of consistency is extended by the product

$$consistency = \prod_{j=0}^{k-1} A(s_j, s_{j+1}) \quad (1)$$

for $s_j < s_{j+1}$ and $j \in \{0,1,2,3,...,k-1\}$ that is

$$A(s_0,s_1)A(s_1,s_2)A(s_2,s_3)\,...A(s_{k-2},s_{k-1})A(s_{k-1},s_k) = A(s_0,s_k) \quad (1a)$$

$$s_0 < s_1 < s_2 < s_3 < ... < s_{k-1} < s_k \quad (2)$$

The no arbitrage condition is a fundamental phenomenon in insurance asset pricing. Since insurance market with no arbitrage opportunities operates such that investors cannot gain bigger than the risk-free rate of return without experiencing any risk, the probability of earning bigger than risk free rate of return should therefore approach zero for zero net investments.

Suppose $A(s_1, s_2)$ satisfies the consistency conditions, then the interest rate is known as compound interest. This justifies application of compound interest in actuarial computations. The nominal interest rate δ is often applied when computing actuarial present value functions in life contingencies especially when cash flows are paid continuously. It is also used as an approximation to interest rates convertible frequently, such as daily conversions. A question yet unanswered in actuarial science is that how many terms of the series

$$\delta = \ln(1+i) = i - \frac{i^2}{2} + \frac{i^3}{3} - \frac{i^4}{4} + ... + (-1)^m \frac{i^m}{m} + ..., \quad (2a)$$

will be sufficient to reasonably estimate the force of interest to the desired degree of accuracy. Although convergence seems to be of concern, but as long as $i < 1$ which is always the case, the above series converges. The existing literatures have tried to develop models for δ and $e^{-\delta t}$ directly using Taylor's series without any successful closed form formula. This has created gaps in literature which are deeply addressed in this paper.

1.1 Theorem 1

Let $A(\zeta)\delta(\zeta)\Delta\zeta$ be the interest earned on the fund $A(\zeta)$ in the small time $\Delta\zeta$. Given that $\pi(\zeta)\Delta\zeta$ is the premium rate received continuously by the life office after time $\Delta\zeta$ such that $1+\delta(\zeta)\Delta\zeta$ is the total interest accumulated on a fund of 1 unit in the small time

$\Delta\zeta$, then at time h, the accumulation function is obtained as

$$A(h)=e^{\int_0^h \delta(u)du}\left(1+\int_0^h \pi(\zeta)e^{-\int_0^\zeta \delta(u)du}d\zeta\right)$$ with the boundary condition $A(0)=1$

Proof

Observe that $\delta(\tau)=\frac{A'(\tau)}{A(\tau)}$ is the instantaneous rate of interest and $A(\zeta)$ is the accumulated fund function at time ζ then $A(\zeta)\delta(\zeta)d\zeta$ will be the infinitesimal interest added at time $d\zeta$ to the fund so that the total interest earned within the infinitesimal time interval $[\zeta,\zeta+\Delta\zeta]$ is $\int_\zeta^{\zeta+\Delta\zeta} A(\tau)\delta(\tau)d\tau$, where

$$\int_\zeta^{\zeta+\Delta\zeta} A(\tau)\delta(\tau)d\tau=\int_\zeta^{\zeta+\Delta\zeta} A(\tau)\frac{A'(\tau)}{A(\tau)}d\tau=\int_\zeta^{\zeta+\Delta\zeta} A'(\tau)d\tau=\int_\zeta^{\zeta+\Delta\zeta} dA(\tau) \tag{3}$$

$$\int_\zeta^{\zeta+\Delta\zeta} A(\tau)\delta(\tau)d\tau=A(\zeta+\Delta\zeta)-A(\zeta) \tag{4}$$

The function $A(\zeta)\delta(\zeta)$ is the interest earned on the accumulated value function $A(\zeta)$ in the continuous time interval $[\zeta,\zeta+\Delta\zeta]$ noting that $A(\zeta)$ is the accumulated value function at time ζ on a sequence of continuous premium cash flows receivable at the rate $\pi(\xi)$ for $0\le\xi\le\zeta$ .Within the infinitesimal time interval $[\zeta,\zeta+\Delta\zeta]$ the total premium earned by the life office will be $\pi(\zeta)\Delta\zeta$, which is presumably obtained at time ζ so that the accumulated value at time $\zeta+\Delta\zeta$ is the sum of the accumulated values at time ζ plus the interest earned on it plus the premium and the interest earned on the premium, hence

$$A(\zeta+\Delta\zeta)=A(\zeta)+A(\zeta)\delta(\zeta)\Delta\zeta+\pi(\zeta)\Delta\zeta\left[1+\delta(\zeta)\Delta\zeta\right] \tag{5}$$

$1+\delta(\zeta)\Delta\zeta$ is the total interest element earned on the premium $\pi(\zeta)$ during the small time $\Delta\zeta$
Therefore, the change in accumulated value function becomes

$$A(\zeta+\Delta\zeta)-A(\zeta)=\pi(\zeta)\Delta\zeta\left[1+\delta(\zeta)\Delta\zeta\right]+A(\zeta)\delta(\zeta)\Delta\zeta \tag{6}$$

Dividing (6) through by $\Delta\zeta$ yields

$$\frac{A(\zeta+\Delta\zeta)-A(\zeta)}{\Delta\zeta}=\pi(\zeta)\left[1+\delta(\zeta)\Delta\zeta\right]+A(\zeta)\delta(\zeta) \tag{7}$$

Taking the limit $\Delta\zeta\to 0$ yields,

$$\lim_{\Delta\zeta\to 0}\left[\frac{A(\zeta+\Delta\zeta)-A(\zeta)}{\Delta\zeta}\right]=\lim_{\Delta\zeta\to 0}\left[\pi(\zeta)\left[1+\delta(\zeta)\Delta\zeta\right]+A(\zeta)\delta(\zeta)\right] \tag{8}$$

$$A'(\zeta)=\pi(\zeta)\left[1+\delta(\zeta)\times 0\right]+A(\zeta)\delta(\zeta) \tag{9}$$

$$A'(\zeta)=\lim_{\Delta\xi\to 0}\left[\pi(\zeta)+A(\zeta)\delta(\zeta)\right] \quad (10)$$

$$A'(\zeta)=\pi(\zeta)+A(\zeta)\delta(\zeta) \quad (10a)$$

$$A'(\zeta)-\delta(\zeta)A(\zeta)=\pi(\zeta) \quad (11)$$

The present value of 1 unit of money due in ζ years is the integrating factor $1\times e^{-\int_0^{\zeta}\delta(u)du}$, use the fact that

$$A'(\zeta)e^{-\int_0^{\zeta}\delta(u)du}-\delta(\zeta)A(\zeta)e^{-\int_0^{\zeta}\delta(u)du}=\pi(\zeta)e^{-\int_0^{\zeta}\delta(u)du} \quad (12)$$

to conclude that

$$\frac{d}{d\zeta}\left(A(\zeta)e^{-\int_0^{\zeta}\delta(u)du}\right)=\pi(\zeta)e^{-\int_0^{\zeta}\delta(u)du} \quad (13)$$

We then integrate with respect to ζ from 0 to h and obtain

$$A(h)e^{-\int_0^{h}\delta(u)du}-A(0)e^{-\int_0^{0}\delta(u)du}=\int_0^h \pi(\zeta)e^{-\int_0^{\zeta}\delta(u)du}d\zeta \quad (14)$$

Then the boundary condition $A(0)=1$, in the theorem yields

$$A(h)e^{-\int_0^{h}\delta(u)du}-1=\int_0^h \pi(\zeta)e^{-\int_0^{\zeta}\delta(u)du}d\zeta \quad (15)$$

Multiply both sides of equation (15) by $e^{\int_0^{h}\delta(u)du}$ yields

$$A(h)=e^{\int_0^{h}\delta(u)du}\left(1+\int_0^h \pi(\zeta)e^{-\int_0^{\zeta}\delta(u)du}d\zeta\right) \quad (16)$$

$$A(h)=e^{\int_0^{h}\delta(u)du}+e^{\int_0^{h}\delta(u)du}\int_0^h \pi(\zeta)e^{-\int_0^{\zeta}\delta(u)du}d\zeta \quad (17)$$

$$A(h)=e^{\int_0^{h}\delta(u)du}+\int_0^h \pi(\zeta)e^{\int_{\zeta}^{h}\delta(u)du}d\zeta \quad (18)$$

This completes the proof

1.2 Theorem 2

Let $A:\mathbf{R}^{+}\to[1,\infty)$ be a monotone increasing right continuous accumulation function with

$$A(0)=1 \tag{18a}$$

Then

$$\frac{A(\zeta+\delta\zeta)-A(\zeta)}{\int_{\zeta}^{\zeta+\delta\zeta}A(0)A(\theta)d\theta}=\delta(\zeta) \tag{19}$$

Proof

A is monotone increasing *iff* $A(x)<A(y)$ for $x<y$ and A is monotone nondecreasing *iff* $A(x)\le A(y)$ for $x<y$

It is necessary to first estimate the denominator. Let $A(\zeta)$ represent the continuously differentiable accumulation and amount functions respectively where $\delta\zeta$ is small time.

because the amount of interest earned by an initial investment of $A(0)$ over $\zeta+\xi$ period is equal to the amount of interest earned if the investment is terminated at the end of ζ periods and the accumulated value at that point is immediately re-invested for an additional ξ periods for $\zeta>0$ and $\xi>0$

The initial investment $A(0)=1$ and $A(A(0))=k\Rightarrow A(1)=k$ for some $k\in$ or $^{+}$

$$1+A(A(0))\ge 2 \tag{20}$$

$$\Rightarrow 1+k\ge 2 \tag{20a}$$

$$\int_{\zeta}^{\zeta+\delta\zeta}A(0)A(\theta)d\theta=\int_{\zeta}^{\zeta+\delta\zeta}A(\theta)d\theta \tag{21}$$

In the integral $J=\int_{x_0}^{x_n}ydx$, we can approximate y by Newton's forward interpolation formula.

$$\int_{x_0}^{x_n}ydx=\int_{x_0}^{x_n}\left[y_0+p\Delta y_0+\frac{p(p-1)}{2}\Delta^2y_0+\frac{p(p-1)(p-2)}{6}\Delta^3y_0+...\right]dx$$
$$=\int_{x_0}^{x_n}y_0dy+\sum_{j=1}^{\infty}{}^{p}C_j\Delta^jy_0 \tag{22}$$

$$\int_{x_0}^{x_n}ydx\cong\int_{x_0}^{x_n}\left[y_0+p\Delta y_0+\frac{p(p-1)}{2}\Delta^2y_0+\frac{p(p-1)(p-2)}{6}\Delta^3y_0\right]dx \tag{22a}$$

Let $x=x_0+ph\Rightarrow dx=hdp$ and note that $x_n=x_0+nh$. When $x=x_0$, $p=0$ and when $x=x_n$

$$x_n = x_0 + ph \Rightarrow x_0 + nh = x_0 + ph \Rightarrow p = n \tag{23}$$

$$\int_{x_0}^{x_n} y dx = h \int_0^n \left[y_0 + p \Delta y_0 + \frac{p(p-1)}{2} \Delta^2 y_0 + \frac{p(p-1)(p-2)}{6} \Delta^3 y_0 + \ldots \right] dp \tag{24}$$

Now truncating at the fourth term yields

$$\int_{x_0}^{x_n} y dx \cong nh \left[y_0 + \frac{n}{2} \Delta y_0 + \frac{n(2n-3)}{12} \Delta^2 y_0 + \frac{n(n-2)^2}{24} \Delta^3 y_0 \right] \tag{25}$$

choose $n = 1; x_n = \zeta + \delta\zeta; x_0 = \zeta$; $y(x_0) = y_0$, so that $h = (\delta\zeta)$, to obtain

$$\int_{x_0}^{x_n} y dx = h \left[y_0 + \frac{1}{2} \Delta y_0 \right] \tag{26}$$

$$\int_{\zeta}^{\zeta+\delta\zeta} A(\theta) d\theta = (\delta\zeta) \left[A(\zeta) + \frac{1}{2} \Delta A(\zeta) \right] \tag{27}$$

$$\int_{\zeta}^{\zeta+\delta\zeta} A(\theta) d\theta = (\delta\zeta) \left[A(\zeta) + \frac{1}{2} \left\{ A(\zeta+1) - A(\zeta) \right\} \right] \tag{28}$$

$$\int_{\zeta}^{\zeta+\delta\zeta} A(\theta) d\theta = (\delta\zeta) \left[A(\zeta) + \frac{1}{2} A(\zeta) A(1) - \frac{1}{2} A(\zeta) \right] \tag{29}$$

$$\int_{\zeta}^{\zeta+\delta\zeta} A(\theta) d\theta = (\delta\zeta) \left[\frac{1}{2} A(\zeta) + \frac{1}{2} A(\zeta) A(1) \right] \tag{30}$$

$$\int_{\zeta}^{\zeta+\delta\zeta} A(\theta) d\theta = (\delta\zeta) A(\zeta) \left[\frac{1}{2} + \frac{1}{2} A(0) A(1) \right] = (\delta\zeta) A(\zeta) \left[\frac{1}{2} + \frac{1}{2} A(A(0)) \right] \tag{31}$$

$$\Rightarrow \int_{\zeta}^{\zeta+\delta\zeta} A(\theta) d\theta = \frac{1}{2} \delta\zeta A(\zeta) \left[1 + A(1) \right] = \frac{1}{2} \delta\zeta A(\zeta) [1 + k] \tag{32}$$

Taking $A(1) = k$,where $k \geq 1$, in (32), we have

$$\int_{\zeta}^{\zeta+\delta\zeta} A(\theta) d\theta = \frac{1}{2} (\delta\zeta) A(\zeta) \times 2 \tag{33}$$

$$\int_{\zeta}^{\zeta+\delta\zeta} A(\theta) d\theta \cong (\delta\zeta) A(\zeta) \tag{34}$$

Observe that $A(\zeta + \delta\zeta) - A(\zeta)$ is the change in accumulation factor, in the elapsed time $\delta\zeta$

Dividing the numerator and denominator of $\frac{A(\zeta+\delta\zeta)-A(\zeta)}{\int_{\zeta}^{\zeta+\delta\zeta} A(\theta)d\theta}$ by $(\delta\zeta)$ and taking limit as $\delta\zeta \to 0$ yields

$$\lim_{\delta\zeta\to 0}\left[\frac{\frac{A(\zeta+\delta\zeta)-A(\zeta)}{\delta\zeta}}{\frac{A(0)}{(\delta\zeta)}\int_{\zeta}^{\zeta+\delta\zeta} A(\theta)d\theta}\right]=\lim_{\delta\zeta\to 0}\frac{\frac{A(\zeta+\delta\zeta)-A(\zeta)}{\delta\zeta}}{\frac{A(0)}{\delta\zeta}\int_{\zeta}^{\zeta+\delta\zeta} A(\theta)d\theta} \tag{35}$$

$$\lim_{\delta\zeta\to 0}\left[\frac{\frac{A(\zeta+\delta\zeta)-A(\zeta)}{\delta\zeta}}{\frac{A(0)}{(\delta\zeta)}\int_{\zeta}^{\zeta+\delta\zeta} A(\theta)d\theta}\right]\cong\lim_{\delta\zeta\to 0}\frac{\frac{A(\zeta+\delta\zeta)-A(\zeta)}{\delta\zeta}}{\frac{A(0)}{(\delta\zeta)}(\delta\zeta)A(\zeta)}=\lim_{\delta\zeta\to 0}\frac{\frac{A(\zeta+\delta\zeta)-A(\zeta)}{\delta\zeta}}{A(0)A(\zeta)} \tag{36}$$

$$\lim_{\delta\zeta\to 0}\left[\frac{\frac{A(\zeta+\delta\zeta)-A(\zeta)}{\delta\zeta}}{\frac{A(0)}{\delta\zeta}\int_{\zeta}^{\zeta+\delta\zeta} A(\theta)d\theta}\right]\cong\frac{A'(\zeta)}{A(\zeta)}=\frac{d\log_e A(\zeta)}{d\zeta}=\delta(\zeta) \tag{37}$$

We deduce that

$$\delta(\zeta)=\frac{d}{d\zeta}\log_e A(\zeta) \tag{37a}$$

This completes the proof

1.3 Accumulation Factors

Insurance firms invest their funds in order to exploit the growth of such funds under the framework of compound interests as time s measured in years goes forward. Let $s_1 < s_2$; the accumulation function $A(s_1, s_2)$ is the accumulation at time s_2 of a unit investment made at time s_1 for a term $(s_2 - s_1)$. We describe the nominal rate of interest $i_h(s)$ per unit time on investment of term h commencing at time s to be the rate of interest such that the effective rate of interest for the period of length h beginning at time s is $hi_h(s)$. Therefore, for all s and all $h>0$, the accumulation function over a unit time of length h is given by

$$A(s,s+h)=1+hi_h(s) \tag{38}$$

Consequently,

$$i_h(s)=\frac{A(s,s+h)-1}{h} \tag{39}$$

By definition $A(s,s)=1$ that is accumulation function is 1 when $h=0$ $A(s,s)=1$. In practice, it is not unreasonable to assume that as h progressively becomes smaller, then

$i_h(s)$ approaches a limiting value which depends on s. Therefore, in life insurance underwriting, it is assumed that there exists a real number $\delta(s)$ such that $\lim_{h\to 0^+} i_h(s) = \delta(s)$. The number $\delta(s)$ is the interest rate intensity per unit time at time s. It defines the nominal rate of interest per unit time at time s convertible momentarily.

$$\lim_{s\to 0^+} i_h(s) = \lim_{s\to 0^+}\left(\frac{A(s,s+h)-1}{h}\right) = \delta(s) \tag{40}$$

The computation of the accumulation function $A(s_1,s_2)$ is immediate, following the specification of the interest rate intensity.

1.4 The Taylor's Series Expansion

The Taylor's theorem describes a technique for estimating actuarial functions. The theorem establishes a Taylor polynomial on the estimation of a differentiable function about a known point. With polynomial estimation, the behaviour of a function can simply be investigated with consequent resulting reduction in calculation. Although the theorem has inspired uncommon actuarial research interests in different domains, its application in the analysis of nominal rates of interest has not been fully explored. The use of Taylor's theorem in financial mathematics enhances the search for unique solutions to interest rate problems. In the sequel, $f \in C^{\infty}$

let

$$\eta(s) = f(x_0 + sh, y_0 + sk), \tag{41}$$

$0 \le s \le 1$,

The equation (41) can be expressed as

$$\eta(s) = f(x(s), y(s)); \forall(s) \tag{42}$$

$$x(s) = x_0 + sh \tag{43}$$

$$y(s) = y_0 + sk \tag{44}$$

Since $0 \le s \le 1$, it follows that

$$\eta(0) = f(x_0, y_0) \tag{45}$$

$$\eta(1) = f(x_0 + h, y_0 + k) \tag{46}$$

It can be inferred from Hongsheng and Bocheng (1986), Zvonimir (1990), Jiguang and Mingyan (2003) and Ma (2023) that the Taylor's series expansion of the function $\eta(1)$ at the point (x_0, y_0) is the Maclaurin's formula at the point 0 when $s=1$ in the univariate function $\eta(s)$

$$\eta(1) = f(0) + f^{(1)}(0) + \frac{f^{(2)}(0)}{2!} + \frac{f^{(3)}(0)}{3!} + \ldots + \frac{f^{(m)}(0)}{m!} + \frac{f^{(m+1)}(\alpha)}{(m+1)!} \tag{47}$$

For $0 < \alpha < 1$. Suppose the function $f(x)$ has m derivatives, then the neighbourhood of x exists and for any x in the neighbourhood, then there is

$$f(x) = f(x_0) + (x-x_0) f^{(1)}(x_0) + \frac{(x-x_0)^2 f^{(2)}(x_0)}{2!} + \frac{(x-x_0)^3 f^{(3)}(x_0)}{3!} + \ldots + \frac{(x-x_0)^m f^{(m)}(x_0)}{m!} + o\left((x-x_0)^m\right) \tag{48}$$

Expanding equation (47) we can define

$$f\left(x_0+h, y_0+k\right)= f\left(x_0, y_0\right)+\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right) f\left(x_0, y_0\right)+\frac{1}{2!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f\left(x_0, y_0\right)$$
$$+\frac{1}{3!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^3 f\left(x_0, y_0\right)+\ldots+\frac{1}{m!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^m f\left(x_0, y_0\right) \quad (49)$$
$$+\frac{1}{(m+1)!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{m+1} f\left(x_0+\alpha h, y_0+\alpha k\right)$$

for $0<\alpha<1$.

$$\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^m f\left(x, y\right)=\sum_{u=0}^{m}\binom{m}{u}h^{m-u}k^u\frac{\partial^m}{\partial x^{m-u}y^u}f\left(x, y\right) \quad (50)$$

$h\neq 0; k\neq 0$

$$\eta(1)= f\left(x_0+h, y_0+k\right) \quad (50a)$$

$$\eta(1)=\sum_{m=0}^{\infty}\sum_{u=0}^{m}\binom{m}{u}h^{m-u}k^u\frac{\partial^m}{\partial x^{m-n}\partial y^u} f\left(x, y\right)\Bigg|_{(x,y)=(x_0,y_0)} \quad (50b)$$

$$\eta(1)= f\left(x_0, y_0\right)+\sum_{m=1}^{\infty}\sum_{u=0}^{m}\binom{m}{u}h^{m-u}k^u\frac{\partial^m}{\partial x^{m-n}\partial y^u} f\left(x, y\right)\Bigg|_{(x,y)=(x_0,y_0)} \quad (50c)$$

$$\eta(1) \square \sum_{m=0}^{\infty}\left[\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{(m)} f\left(x, y\right)\right]_{(x,y)=(x_0,y_0)} \quad (50d)$$

where

$$\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{(m)}=\sum_{u=0}^{m}\binom{m}{n}h^{m-n}k^u\frac{\partial^m}{\partial x^{m-n}\partial y^u} f\left(x, y\right)\Bigg|_{(x,y)=(x_0,y_0)} \quad (50e)$$

Now for $0<\alpha<1$, sufficiently small $f\left(x_0+h, y_0+k\right)$ is approximable to

$$\sum_{j=0}^{m}\sum_{u=0}^{j}\binom{j}{u}h^{j-u}k^u\frac{\partial^j}{\partial x^{j-u}\partial y^u} f\left(x, y\right)\Bigg|_{(x,y)=(x_0,y_0)}+$$
$$\frac{1}{(m+1)!}\left[\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{(m+1)} f\left(x+\alpha h, y+\alpha k\right)\right]_{(x,y)=(x_0,y_0)} \quad (50f)$$

2.0 Material and Methods

The accumulation function as a function of two different investment times satisfies the consistency conditions and its behaviour can be investigated to obtain the force of interest. As a result, the Taylor's series expansion in two dimensions will be deployed to derive the corresponding interest rate intensity. The derivations obtained here are driven from the theoretical perspectives and are meant to set aside the comparatively intractable process of deterministic interest rate intensity computations. In order to ensure that the Taylor's expansion falls in line with the continuous time the accumulation function, the following assumptions were made. The accumulation function (i) $A(s,t)\in C^2\left(R^2\right)$, this ensures the existence of first order and second order partial derivatives and justifies the employment the two-dimensional Taylor's expansion, (ii) the function $A(s,t)$ satisfies $A(s,t)=A(s,u)A(u,t); \quad s\le u\le t$ (iii) the function $A(s,t)$ satisfies the normalisation equation $A(s,s)=1$. This is the initialisation of the accumulation function employed to model the instantaneous interest rate, (iv) the function $A(s,t)$ satisfies the positivity condition $A(s,t)\ge 0; \quad s\le t$ (v) the derivation assume that the function $A(s,t)$ defined over a short period of time interval Δ can be estimated by a Taylor's series expansion around the point (s,s) that is

$$A(s,s+\Delta)\cong A(s,s)+\frac{\partial}{\partial s_2}A(s,s)\Delta+o(\Delta); \quad \Delta\to 0 \tag{50g}$$

$$A(s,s+\Delta)\cong 1+\frac{\partial}{\partial s_2}A(s,s)\Delta+o(\Delta); \quad \Delta\to 0 \tag{50h}$$

$$\delta(s)=\frac{\partial}{\partial s_2}\log_e A(s,s_2)_{s_2=s}; \quad s_2=s+\Delta \tag{50i}$$

Analytically, the following diagnostics are analytically valid: (i) the interest rate intensity yields the function $A(s,t)$ which satisfies consistency and initialization conditions

$$A(s,t)=A(s,u)A(u,t); \quad A(s,s)=1 \tag{50j}$$

(ii) the function $A(s,t)$ is exponentially reconstructed as

$$A(s,t)=\exp\left(\int_s^t \delta(u)\,du\right) \tag{50k}$$

(iii) the limiting value of the interest rate intensity exists and is unique

$$\delta(s)=\lim_{\Delta\to 0}\frac{A(s,s+\Delta)-1}{\Delta} \tag{50l}$$

(iv) because the function $A(s,t)$ is two dimensional in form, limiting the Taylor's series expansion to the direction $(0,\Delta)$ yields a derivative which falls in line with approaching the diagonal $s_1=s_2$ from admissible directions and hence the interest rate intensity $\delta(s)$ is not a function subject to parametrization, (v) the instantaneous rate of interest is finite and yields accumulation function $A(s,t)>0$ aligning with the continuous compounding interpretations.

Let $U\subset\mathbf{R}^2$ be open and suppose the actuarial accumulation function $A(s,t)$, $A:G\to\mathbf{R}$ defines accumulation function and satisfies continuous mixed partial derivatives up to order κ on U. Then there exists an open neighborhood $U\subset\mathbf{R}^2$ of (α_0,β_0). Since $U\subset\mathbf{R}^2$ is compact, there exists a rectangle $G=[A,B]\times[C,D]; \quad \exists: K\subset G\subset U$ and simply follows from the compactness of K and openness of U. It is sufficient to show the integrability over G. Observe that the integrability over G implies integrability over its subset $K\subset G$

Fixing the integers $r,s\ge 0$ are non-negative integers with $r+s\le\kappa$ yields

$$f(\alpha,\beta) = D^{r,s}A(\alpha,\beta) = \frac{\partial^{r+s}}{\partial\alpha^r\partial\beta^s}A(\alpha,\mu)A(\mu,\beta) = \frac{\partial^{r+s}}{\partial\alpha^r\partial\beta^s}A(\alpha,\beta); \quad \alpha,\beta \in \mathbf{R} \tag{50m}$$

is continuous on U and hence continuous over the closed rectangle $G \subset U$. Since G is compact and f is continuous over G, the Heine-Cantor theorem can be invoked that is a continuous function on a compact set is uniformly continuous. Therefore, for every $\in > 0$, there exists $\overline{\delta} > 0\ \exists$:

$$\left|(\alpha,\beta) - \left(\overline{\alpha},\overline{\beta}\right)\right| < \overline{\delta} \Rightarrow \left|f(\alpha,\beta) - f\left(\overline{\alpha},\overline{\beta}\right)\right| < \in \tag{50n}$$

We can construct a partition $\mathbf{P}$ of G into finitely many auxiliary rectangles $G_{mn} = [\alpha_{m-1},\alpha_m]\times[\beta_{m-1},\beta_m]$ such that the diameter of each sub-rectangle satisfies the inequality $diameter(G_{mn}) < \overline{\delta}$. For each auxiliary rectangle G_{mn}, we now define the supremum and infimum of f in equation (50o) as follows

$$\begin{cases} N_{mn} = \sup\limits_{(\alpha,\beta)\in G_{mn}} f(\alpha,\beta) \\ n_{mn} = \inf\limits_{(\alpha,\beta)\in G_{mn}} f(\alpha,\beta) \end{cases} \tag{50o}$$

Since $diameter(G_{mn}) < \overline{\delta}$ and $diameter(G_{mn})$ is uniform continuity means that,

$$\sup_{(\alpha,\beta)\in G_{mn}} f(\alpha,\beta) - \inf_{(\alpha,\beta)\in G_{mn}} f(\alpha,\beta) < \in \tag{50p}$$

$$N_{mn} - n_{mn} < \in \tag{50q}$$

Define the upper and lower sums

$$u(f,\mathbf{P}) = \sum_{m,n} N_{mn} \times Area(G_{mn}) \tag{50r}$$

$$\Rightarrow l(f,\mathbf{P}) = \sum_{m,n} n_{mn} \times Area(G_{mn}) \tag{50s}$$

$$\Rightarrow u(f,\mathbf{P}) - l(f,\mathbf{P}) = \sum_{m,n} N_{mn} \times Area(G_{mn}) - \sum_{m,n} n_{mn} \times Area(G_{mn}) \tag{50t}$$

$$\Rightarrow u(f,\mathbf{P}) - l(f,\mathbf{P}) = \sum_{m,n} (N_{mn} - n_{mn}) \times Area(G_{mn}) \tag{50u}$$

Since $(N_{mn} - n_{mn}) < \in$ for all sub-rectangles

$$u(f,\mathbf{P})-l(f,\mathbf{P})=\sum_{m,n}\in\times Area(G_{mn}) \tag{50v}$$

Observe that $Area(G)=\sum_{m,n}Area(G_{mn})$

$$\Rightarrow u(f,\mathbf{P})-l(f,\mathbf{P})=\in\times Area(G) \tag{50w}$$

Then $\exists$ a bounded function which is Riemann integrable over G if for every $\theta>0$ and there exists a partition $\mathbf{P}$ such that

$$u(f,\mathbf{P})-l(f,\mathbf{P})<\theta \tag{50x}$$

Given any $\theta>0$, we can take $\in$ where $\in=\dfrac{\theta}{Area(G)}$. Consequently, the partition constructed above satisfies

$$\Rightarrow u(f,\mathbf{P})-l(f,\mathbf{P})<\theta \tag{50y}$$

However, f is Riemann integrable on G. Since $K\subset G$ and f is Riemann integrable over G, then the restriction of f to K defined as $f=\left.\dfrac{\partial^{r+s}}{\partial\alpha^r\partial\beta^s}A(\alpha,\beta)\right|_G$ is integrable over K. For every compact set $K\subset U$ and for each pair of integers (r,s) with $r+s\le\kappa$, the mixed partial derivative

$$\Rightarrow D^{r,s}A(\alpha,\beta)=\frac{\partial^{r+s}}{\partial\alpha^r\partial\beta^s}A(\alpha,\beta) \tag{50z}$$

is Riemann integrable over K
Define

$$A:\mathbf{R}^2\to\mathbf{R} \tag{51a}$$

and $(s_0,t_0)\in\mathbf{R}^2$ by

$$A(s,t)=\exp\left(\int_s^t\delta(u)\,du\right) \tag{52}$$

where s is an arbitrary time of investment of a life insurance fund and t is the time of maturity. Then for $s<t$; $A(s,t)$ is the income which the life insurance office receives at time t in return for a life fund investment of 1 at time s. Consider the investment times, $\{s,u,t\}$ where $s<u<t$ and such that the accumulation function $A(s,t)$ satisfies the consistency conditions

$$A(s,t)=A(s,u)A(u,t) \tag{52a}$$

with the initial condition $A(0,0)=A(0)=1$, where the initial investment at time $s=0$ is 1. The condition ascertains that accumulating life insurance fund over successive intervals of time is equivalent to accumulating it over the whole interval at once.

Define

$$A(s_0,s)=G(s) \tag{53}$$

s_0 is the initial time of investment of life fund

$$\delta(s)=\lim_{h\to 0}\frac{A(s,s+h)-A(0,0)}{h} \tag{54}$$

Multiply and divide the RHS of (54) by $A(s_0,s)$ to get

$$\delta(s)=\frac{1}{A(s_0,s)}\lim_{h\to 0}\frac{A(s_0,s)A(s,s+h)-A(s_0,s)A(0,0)}{h} \tag{55}$$

It follows from the consistency principle that

$$\delta(s)=\frac{1}{A(s_0,s)}\lim_{h\to 0}\frac{A(s_0,s+h)-A(s_0,s)}{h} \tag{56}$$

$$\delta(s)=\frac{1}{G(s)}\lim_{h\to 0}\frac{G(s+h)-G(s)}{h}=\frac{G'(s)}{G(s)}\Rightarrow G'(s)=\delta(s)G(s) \tag{57}$$

$$G(s)=e^{\int_{s_0}^{s}\delta(u)du} \tag{58}$$

$$A(s,u)=e^{\int_{s}^{u}\delta(\xi)d\xi};\quad A(u,t)=e^{\int_{u}^{t}\delta(\xi)d\xi} \tag{59}$$

$$A(s,u)A(u,t)=\exp\left(\int_s^u\delta(\xi)d\xi\right)\times\exp\left(\int_u^t\delta(\xi)d\xi\right)=e^{\int_u^t\delta(\xi)d\xi-\int_u^s\delta(\xi)d\xi}=$$
$$\exp\left(\int_s^t\delta(\xi)d\xi\right)=A(s,t)=\frac{A(u,t)}{A(u,s)} \tag{60}$$

$$A(s,t)=\frac{G(t)}{G(s)} \tag{61}$$

Based on the behavior of the accumulation function**,** the amount of life fund invested at times $s+\Delta s$ and $s-\Delta s$ are defined by $A(s+\Delta s)$ and $A(s-\Delta s)$ respectively. There can be expanded into Taylor series at s_0 as follows:

$$A(s_0+\Delta s)=A(s_0)+(\Delta s)A^{(1)}(s_0)+(\Delta s)^2\frac{A^{(2)}(s_0)}{2!}+(\Delta s)^3\frac{A^{(3)}(s_0)}{3!}$$
$$+(\Delta s)^4\frac{A^{(4)}(s_0)}{4!}...+(\Delta s)^j\frac{A^{(j)}(s_0)}{j!}+\int_{s_0}^{s}\frac{(\Delta s)^j A^{(j+1)}(\xi)}{j!}d\xi \tag{62}$$

$$A(s_0-\Delta s)=A(s_0)-(\Delta s)A^{(1)}(s_0)+(\Delta s)^2\frac{A^{(2)}(s_0)}{2!}-(\Delta s)^3\frac{A^{(3)}(s_0)}{3!}$$
$$+(\Delta s)^4\frac{A^{(4)}(s_0)}{4!}+...+(-1)^j(\Delta s)^j\frac{A^{(j)}(s_0)}{j!}+\int_{s_0}^{s}\frac{(\Delta s)^j A^{(j+1)}(\xi)}{j!}d\xi \tag{63}$$

If the insurance fund satisfies the consistency conditions in the interval $[s,t]$ and s is fixed, then the size of the fund together with investment returns at time t evaluated at the point (s_0,t_0) is estimated as a two point Taylor's series. $A(s,t)$ is assumed to have continuous partial derivatives of all orders $A\in C^{\infty}$

$$\begin{aligned}A(s,t)&=A(s_0,t_0)+(s-s_0)\left\{\frac{\partial A(s_0,t_0)}{\partial s}\right\}+(t-t_0)\left\{\frac{\partial A(s_0,t_0)}{\partial t}\right\}+\frac{(s-s_0)^2}{2!}\left\{\frac{\partial^2 A(s_0,t_0)}{\partial s^2}\right\}\\&+\frac{(t-t_0)^2}{2!}\left\{\frac{\partial^2 A(s_0,t_0)}{\partial t^2}\right\}+(s-s_0)(t-t_0)\left\{\frac{\partial^2 A(s_0,t_0)}{\partial s\partial t}\right\}+\frac{(s-s_0)^3}{3!}\left\{\frac{\partial^3 A(s_0,t_0)}{\partial s^3}\right\}\\&+\frac{(t-t_0)^3}{3!}\left\{\frac{\partial^3 A(s_0,t_0)}{\partial t^3}\right\}+\frac{(s-s_0)^2(t-t_0)}{2}\left\{\frac{\partial^3 A(s_0,t_0)}{\partial s^2\partial t}\right\}\\&+\frac{(s-s_0)(t-t_0)^2}{2}\left\{\frac{\partial^3 A(s_0,t_0)}{\partial s\partial t^2}\right\}+O\left(|s-s_0|^4\right)+O\left(|t-t_0|^4\right)\end{aligned} \tag{64}$$

is a third order approximation

Letting $s=s_0+\Delta s$ and $t=t_0+\Delta t$ in (64) and suppose Δs and Δt be small time increment in s and t respectively, then

$$\begin{aligned}A(s_0+\Delta s,t_0+\Delta t)&=A(s_0,t_0)+(\Delta s)\left\{\frac{\partial A(s_0,t_0)}{\partial s}\right\}+(\Delta t)\left\{\frac{\partial A(s_0,t_0)}{\partial t}\right\}+\frac{(\Delta s)^2}{2!}\left\{\frac{\partial^2 A(s_0,t_0)}{\partial s^2}\right\}\\&+\frac{(\Delta t)^2}{2!}\left\{\frac{\partial^2 A(s_0,t_0)}{\partial t^2}\right\}+(\Delta s)(\Delta t)\left\{\frac{\partial^2 A(s_0,t_0)}{\partial s\partial t}\right\}+\frac{(\Delta s)^3}{3!}\left\{\frac{\partial^3 A(s_0,t_0)}{\partial s^3}\right\}\\&+\frac{(\Delta t)^3}{3!}\left\{\frac{\partial^3 A(s_0,t_0)}{\partial t^3}\right\}+\frac{(\Delta s)^2(\Delta t)}{2}\left\{\frac{\partial^3 A(s_0,t_0)}{\partial s^2\partial t}\right\}+\frac{(\Delta s)(\Delta t)^2}{2}\left\{\frac{\partial^3 A(s_0,t_0)}{\partial s\partial t^2}\right\}\\&+\frac{1}{n!}\sum_{i+j=n+1}\frac{k^i h^j}{i!\,j!}\frac{\partial^{i+j}A(\xi,\eta)}{\partial s^i\partial t^j}\end{aligned} \tag{65}$$

$$\begin{aligned}A(s_0-\Delta s,t_0-\Delta t)&=A(s_0,t_0)-(\Delta s)\left\{\frac{\partial A(s_0,t_0)}{\partial s}\right\}-(\Delta t)\left\{\frac{\partial A(s_0,t_0)}{\partial t}\right\}+\frac{(\Delta s)^2}{2!}\left\{\frac{\partial^2 A(s_0,t_0)}{\partial s^2}\right\}\\&+\frac{(\Delta t)^2}{2!}\left\{\frac{\partial^2 A(s_0,t_0)}{\partial t^2}\right\}+(\Delta s)(\Delta t)\left\{\frac{\partial^2 A(s_0,t_0)}{\partial s\partial t}\right\}-\frac{(\Delta s)^3}{3!}\left\{\frac{\partial^3 A(s_0,t_0)}{\partial s^3}\right\}-\frac{(\Delta t)^3}{3!}\left\{\frac{\partial^3 A(s_0,t_0)}{\partial t^3}\right\}\\&-\frac{(\Delta s)^2(\Delta t)}{2}\left\{\frac{\partial^3 A(s_0,t_0)}{\partial s^2\partial t}\right\}-\frac{(\Delta s)(\Delta t)^2}{2}\left\{\frac{\partial^3 A(s_0,t_0)}{\partial s\partial t^2}\right\}+\frac{1}{n!}\sum_{i+j=n+1}\frac{k^i h^j}{i!\,j!}\frac{\partial^{i+j}A(\xi,\eta)}{\partial s^i\partial t^j}\end{aligned} \tag{66}$$

For some point (ξ,η) on the line segment joining (s_0,t_0) and $(s_0+\Delta s,t_0+\Delta t)$. The upper limit of the summation in remainder R_n is the total order of derivatives $i+j=n+1$. This implies the sum includes all partial derivatives of order exactly $n+1$ and not higher. The interest rate intensity at the points $\delta(s_0,t_0)$ is derived by deploying the partial derivatives of $\delta(s,t)$. The time rate at which the interest intensity changes with respect to s is defined as the partial derivative of $\delta(s,t)$ with respect to s

$$\left.\frac{\partial\delta(s,t)}{\partial s}\right|_{(s_0,t_0)}=\lim_{\Delta s\to 0}\frac{\delta(s_0+\Delta s,t_0)-\delta(s_0,t_0)}{\Delta s} \tag{67}$$

Similarly, the time rate at which the interest intensity changes with respect to t is given by

$$\left.\frac{\partial\delta(s,t)}{\partial t}\right|_{(s_0,t_0)}=\lim_{\Delta t\to 0}\frac{\delta(s_0,t_0+\Delta t)-\delta(s_0,t_0)}{\Delta t} \tag{68}$$

Define

$$\delta(s_0+\Delta s,t_0)=\frac{1}{A(s_0+\Delta s,t_0)}\frac{\partial A(s_0+\Delta s,t_0)}{\partial s} \tag{69}$$

$$\delta(s_0,t_0+\Delta t)=\frac{1}{A(s_0,t_0+\Delta t)}\frac{\partial A(s_0,t_0+\Delta t)}{\partial t} \tag{70}$$

Now set $\Delta s=0$ in (69) to obtain

$$\delta_s(s_0,t_0)=\frac{1}{A(s_0,t_0)}\lim_{\substack{s\to s_0\\ t\to t_0}}\left\{\frac{\partial A(s,t)}{\partial s}\right\}=\frac{1}{A(s_0,t_0)}\frac{\partial A(s_0,t_0)}{\partial s} \tag{71}$$

Now set $\Delta t=0$ in (70) to obtain

$$\delta_t(s_0,t_0)=\frac{1}{A(s_0,t_0)}\frac{\partial A(s_0,t_0)}{\partial t} \tag{72}$$

Taking the limit in (65) as $\Delta t\to 0$ yields

$$\begin{aligned}&A(s_0+\Delta s,t_0)=\\&A(s_0,t_0)+(\Delta s)\left\{\frac{\partial A(s_0,t_0)}{\partial s}\right\}+\frac{(\Delta s)^2}{2!}\left\{\frac{\partial^2 A(s_0,t_0)}{\partial s^2}\right\}+\frac{(\Delta s)^3}{3!}\left\{\frac{\partial^3 A(s_0,t_0)}{\partial s^3}\right\}+\dots\\&=A(s_0,t_0)+\sum_{i=1}^{\infty}\frac{(\Delta s)^i}{i!}\left\{\frac{\partial^i A(s_0,t_0)}{\partial s^i}\right\}\end{aligned} \tag{73}$$

Take the limit (65) again and as $\Delta s\to 0$ to obtain

$$\begin{aligned}A(s_0,t_0+\Delta t)&=A(s_0,t_0)+(\Delta t)\left\{\frac{\partial A(s_0,t_0)}{\partial t}\right\}+\frac{(\Delta t)^2}{2!}\left\{\frac{\partial^2 A(s_0,t_0)}{\partial t^2}\right\}\\&+\frac{(\Delta t)^3}{3!}\left\{\frac{\partial^3 A(s_0,t_0)}{\partial t^3}\right\}+\dots\cong A(s_0,t_0)+\sum_{i=1}^{\infty}\frac{(\Delta s)^i}{i!}\left\{\frac{\partial^i A(s_0,t_0)}{\partial t^i}\right\}\end{aligned} \tag{74}$$

Pass through the limit (66) as $\Delta t\to 0$ to get

$$A\left(s_0-\Delta s,t_0\right)=A\left(s_0,t_0\right)-\left(\Delta s\right)\left\{\frac{\partial A\left(s_0,t_0\right)}{\partial s}\right\}+\frac{\left(\Delta s\right)^2}{2!}\left\{\frac{\partial^2 A\left(s_0,t_0\right)}{\partial s^2}\right\}$$
$$-\frac{\left(\Delta s\right)^3}{3!}\left\{\frac{\partial^3 A\left(s_0,t_0\right)}{\partial s^3}\right\}+\ldots \tag{75}$$

The limit in (66) and tend $\Delta s\to 0$ is the following

$$A\left(s_0,t_0-\Delta t\right)$$
$$=A\left(s_0,t_0\right)-\left(\Delta t\right)\left\{\frac{\partial A\left(s_0,t_0\right)}{\partial t}\right\}+\frac{\left(\Delta t\right)^2}{2!}\left\{\frac{\partial^2 A\left(s_0,t_0\right)}{\partial t^2}\right\}-\frac{\left(\Delta t\right)^3}{3!}\left\{\frac{\partial^3 A\left(s_0,t_0\right)}{\partial t^3}\right\}+\ldots \tag{76}$$

Differentiate (73) with respect to s yields

$$\frac{\partial A\left(s_0+\Delta s,t_0\right)}{\partial s}=\left\{\frac{\partial A\left(s_0,t_0\right)}{\partial s}\right\}+\left(\Delta s\right)\left\{\frac{\partial^2 A\left(s_0,t_0\right)}{\partial s^2}\right\}+\frac{\left(\Delta s\right)^2}{2!}\left\{\frac{\partial^3 A\left(s_0,t_0\right)}{\partial s^3}\right\}$$
$$+\frac{\left(\Delta s\right)^3}{3!}\left\{\frac{\partial^4 A\left(s_0,t_0\right)}{\partial s^4}\right\}+\ldots\cong\sum_{i=0}^{\infty}\frac{\left(\Delta s\right)^i}{i!}\left\{\frac{\partial^{i+1} A\left(s_0,t_0\right)}{\partial s^{i+1}}\right\} \tag{77}$$

Differentiate (74) with respect to t yields

$$\frac{\partial A\left(s_0,t_0+\Delta t\right)}{\partial t}=\left\{\frac{\partial A\left(s_0,t_0\right)}{\partial t}\right\}+\left(\Delta t\right)\left\{\frac{\partial^2 A\left(s_0,t_0\right)}{\partial t^2}\right\}+\frac{\left(\Delta t\right)^2}{2!}\left\{\frac{\partial^3 A\left(s_0,t_0\right)}{\partial t^3}\right\}$$
$$+\frac{\left(\Delta t\right)^3}{3!}\left\{\frac{\partial^4 A\left(s_0,t_0\right)}{\partial t^4}\right\}\cong\sum_{i=0}^{\infty}\frac{\left(\Delta t\right)^i}{i!}\left\{\frac{\partial^{i+1} A\left(s_0,t_0\right)}{\partial t^{i+1}}\right\} \tag{78}$$

Plugging (73) and (77) into (69), result in cases where the initial time of investing insurance fund is marginally increased:

$$\delta\left(s_0+\Delta s,t_0\right)=\frac{\sum_{i=0}^{\infty}\frac{\left(\Delta s\right)^i}{i!}\left\{\frac{\partial^{i+1} A\left(s_0,t_0\right)}{\partial s^{i+1}}\right\}}{A\left(s_0,t_0\right)+\sum_{i=1}^{\infty}\frac{\left(\Delta s\right)^i}{i!}\left\{\frac{\partial^{i} A\left(s_0,t_0\right)}{\partial s^{i}}\right\}} \tag{79}$$

Similarly, put (74) and (78) into (70) to obtain cases where the final time of receiving the insurance fund is marginally increased.

$$\delta\left(s_0,t_0+\Delta t\right)=\frac{\sum_{i=0}^{\infty}\frac{\left(\Delta t\right)^i}{i!}\left\{\frac{\partial^{i+1} A\left(s_0,t_0\right)}{\partial t^{i+1}}\right\}}{A\left(s_0,t_0\right)+\sum_{i=1}^{\infty}\frac{\left(\Delta s\right)^i}{i!}\left\{\frac{\partial^{i} A\left(s_0,t_0\right)}{\partial t^{i}}\right\}} \tag{80}$$

Suppose $A\left(s,t\right)$ is smooth in t and we want to want to expand $A\left(s,t\right)$ about $t=s$, then

$$A\left(s,t\right)=A\left(s,s\right)+\frac{\partial A\left(s,s\right)\left(t-s\right)}{\partial t}+\frac{1}{2!}\frac{\partial^2 A\left(s,s\right)\left(t-s\right)^2}{\partial t^2}\ldots \tag{80a}$$

$$A\left(s,s\right)=1 \tag{80b}$$

and

$$\frac{\partial}{\partial t}A(s,s)=\delta(s) \tag{80c}$$

because

$$A(s,t)=\exp\left(\int_s^t \delta(u)\,du\right)\Rightarrow \frac{d}{dt}A(s,t)=\delta(t)A(s,t) \tag{80d}$$

$$\frac{\partial^2 A(s,s)}{\partial t^2}=\delta'(s)A(s,s)+\delta(s)A'(s,s) \tag{80e}$$

$$\frac{\partial^2 A(s,s)}{\partial t^2}=\delta'(s)A(s,s)+\delta(s)\delta(s)A(s,s) \tag{80f}$$

$$\frac{\partial^2 A(s,s)}{\partial t^2}=\delta'(s)+\left(\delta(s)\right)^2 \tag{80g}$$

Hence,

$$A(s,t)=1+(t-s)\delta(s)+\frac{1}{2!}\left(\delta'(s)+\left(\delta(s)\right)^2\right)(t-s)^2+\ldots \tag{80h}$$

The goal is now to show the consistency condition that

$$A(s,t)=A(s,u)A(u,t) \tag{80i}$$

Now, for $s<u<t$, let $\varepsilon=u-s$ and $\sum=t-u$, therefore, $\varepsilon+\sum=t-s$

$$A(s,u)=1+\delta(s)\varepsilon+\frac{1}{2!}\left(\delta'(s)+\left(\delta(s)\right)^2\right)\varepsilon^2+\ldots \tag{80j}$$

Expand around $t=u$

$$A(u,t)=1+\delta(u)\Sigma+\frac{1}{2!}\left(\delta'(u)+\left(\delta(u)\right)^2\right)\Sigma^2+\ldots \tag{80k}$$

Multiply $A(s,u)A(u,t)$ using product of Taylor Series only up to second order for brevity

$$A(s,t)=\left[1+\delta(s)\varepsilon+\frac{1}{2!}\left(\delta'(s)+\left(\delta(s)\right)^2\right)\varepsilon^2+\ldots\right]\times\left[1+\delta(u)\Sigma+\frac{1}{2!}\left(\delta'(s)+\left(\delta(s)\right)^2\right)\Sigma^2+\ldots\right] \tag{80l}$$

$$Constant=1\times1=1 \tag{80m}$$

$$Linear=\delta(s)\varepsilon+\delta(u)\Sigma \tag{80n}$$

$$Quadratic=\frac{1}{2!}\left(\delta'(s)+\left(\delta(s)\right)^2\right)\varepsilon^2+\frac{1}{2!}\left(\delta'(u)+\left(\delta(u)\right)^2\right)\Sigma^2+\varepsilon\delta(s)\Sigma\delta(s) \tag{80o}$$

$$A(s,t)=1+\delta(s)\varepsilon+\delta(u)\Sigma+\frac{1}{2!}\left(\delta'(s)+\left(\delta(s)\right)^2\right)\varepsilon^2+\frac{1}{2!}\left(\delta'(u)+\left(\delta(u)\right)^2\right)\Sigma^2 +\delta(s)\delta(u)\varepsilon\Sigma+\ldots \tag{80p}$$

Expand $A(s,t)$ directly using $t=s+\varepsilon+\sum$ and equation (80h) yields

$$A(s,t)\cong 1+\delta(s)(\varepsilon+\Sigma)+\frac{1}{2!}\left(\delta'(s)+\left(\delta(s)\right)^2\right)(\varepsilon+\Sigma)^2\ldots \tag{80q}$$

$$(\varepsilon+\Sigma)^2=\varepsilon^2+2\varepsilon\Sigma+\Sigma^2 \tag{80r}$$

$$A(s,t)=1+\delta(s)(\varepsilon+\Sigma)+\frac{1}{2!}\left(\delta'(s)+(\delta(s))^2\right)\left(\varepsilon^2+2\varepsilon\Sigma+\Sigma^2\right)+\ldots \tag{80s}$$

$$A(s,t)=1+\varepsilon\delta(s)+\Sigma\delta(s)+\frac{1}{2!}\varepsilon^2\left(\delta'(s)+(\delta(s))^2\right)+\varepsilon\Sigma\left(\delta'(s)+(\delta(s))^2\right)+\frac{1}{2!}\Sigma^2\left(\delta'(s)+(\delta(s))^2\right)+\ldots \tag{80t}$$

Comparing (80p) and (80t) yields

$$\delta(u)=\delta(s)+\delta'(s)\varepsilon+\ldots \tag{80u}$$

$$\Sigma\delta(u)=\Sigma\delta(s)+\Sigma\varepsilon\delta'(s)+\ldots \tag{80v}$$

$$\varepsilon\Sigma\delta(u)\delta(s)=\varepsilon\delta(\delta(s))^2+\varepsilon^2\Sigma\delta(s)\delta'(s)+\ldots \tag{80w}$$

So when multiplying out $A(s,u)A(u,t)$, the terms up to the second order in ε, Σ and $\varepsilon\Sigma$ match the Taylor's series expansion of $A(s,t)$ directly. Therefore,

$$A(s,t)=A(s,u)A(u,t)+o\left(\varepsilon^2+\varepsilon\Sigma+\Sigma^2\right) \tag{80x}$$

and hence the consistency condition follows in (80y)

$$A(s,t)=A(s,u)A(u,t) \tag{80y}$$

Unlike the current derivations which employs a one-dimensional technique by expanding along the final time or initial times, this paper employs two-dimensional Taylor's series expansion around $s_1=s_2$. This yields rigorous derivations of the interest rate intensity while explicitly incorporating relationship between the initial and final time horizon. The intensity was derived analytically from the accumulation function to overcome the problem of estimating the parameters in a defined parametric form. This generalises the basic actuarial principles which may be extended to time varying accumulation functions. The work specifies the initialisation $A(s,s)=1$ to ascertain that an analytically and actuarially consistent point is achieved in the expansion. The derivation was justified by a sensitivity analysis of the instantaneous interest rate over interest rate regimes $6\%-10\%$ yielding robust rates which are conspicuously unavailable in actuarial estimations.

In life insurance valuation, interest rate is assumed to be constant for the following reasons (i) constant interest rates is a common assumption in actuarial modeling and life insurance valuation, as they offer a practical approach that aligns with industry standards. In practice, using a constant force of interest is a simplifying assumption that many financial models depend on. It allows consistency calculations across different insurance products and policies. This assumption provides a background for deriving financial models that are broadly applicable within the actuarial community, (ii) the life insurance industry typically deals with long-term liabilities, where interest rates are assumed to be stable over the life of a policy. Life insurance contracts, such as whole life or term life insurance policies, often span several decades. A constant force of interest is used to represent a stable economic environment where interest rates do not fluctuate dramatically over the long horizon. This assumption overcomes the need to forecast future interest rates and helps actuaries focus on other aspects of the valuation, such as mortality rates and policyholder behavior, (iii) long-term financial forecasting in life insurance can be highly uncertain and introducing time-varying interest rates adds complexity without significantly improving the accuracy of projections. While interest rates do fluctuate over time in reality, introducing a variable force of interest into long-term life insurance models would significantly complicate the valuation process. Given the implicit uncertainty in future interest rate movements, using a constant force of interest helps simplify the assumptions while still providing reasonable approximations for the purpose of pricing insurance products and determining reserves, (iv) in certain economic conditions, a stable and predictable interest rate environment is assumed for regulatory and pricing purposes. Many life insurance models are created under regulatory frameworks that assume relatively stable interest rates to ensure solvency and adequate reserves. Regulators often require insurers to use conservative assumptions about interest rates to avoid underestimating future liabilities. A constant force of interest allows for standardized valuations across insurers and is in line with the conservative nature of actuarial pricing, (v) the use of a constant force of interest overcomes the sensitivity of life insurance pricing to volatile or unpredictable interest rate changes. Life insurance companies face significant long-term risks due to fluctuations in interest rates, but by assuming a constant force of interest, actuaries can focus on other important risks, such as mortality and morbidity. This assumption provides a buffer against extreme scenarios, such as interest rate shocks or severe market volatility, allowing insurers to maintain a stable pricing framework, (vi) in life insurance valuation, the use of a constant force of interest allows for consistency in actuarial assumptions and methodologies. Many life insurance valuations rely on assumptions about mortality, expenses, and investment returns. Using a constant force of interest ensures that this assumption is compatible with other assumptions, such as mortality rates or policyholder behavior. It provides a straightforward way to harmonize the modeling of all these factors, making the valuation process more tractable and consistent across different policies and products, (vii) a constant force of interest simplifies the discounting process of future cash flows in insurance product pricing.

In life insurance valuation, future premiums, benefits, and claims need to be discounted to their present value. Using a constant force of interest δ, the discount factor becomes $e^{-\delta}$, a simple and easily computable expression. This helps actuaries to quickly determine the present value of future liabilities, such as death benefits or annuity payouts, with fewer assumptions about the changing rate of return over time. Assuming a constant force of interest in life insurance valuation helps simplify the complex process of pricing and reserving for long-term insurance products. It allows actuaries to focus on core factors like mortality rates, premium payments, and policyholder behavior, while reducing the computational challenges associated with variable interest rates. This assumption reflects a practical, industry-standard approach to ensuring solvency and managing risk in the life insurance sector. While it may not perfectly capture real market interest rate fluctuations, it offers a reasonable approximation that provides clarity and consistency in long-term financial planning.

let $\delta:[0,\infty)\to\mathbf{R}^{+}$ be the interest rate intensity**.** Evidently, a unit of investment over a period of length $\frac{1}{m}$ will produce a return of $\left(1+\frac{i^{(m)}}{m}\right)^{m}$ and therefore, equations (39), (40) and the consistency conditions imply that

$$i_h(s)=\frac{A(s,s+h)-1}{1}=A(s,s+h)-1 \tag{81}$$

$$i_1(0)=A(0,1)-1 \tag{81a}$$

Consequently,

$$i_h(s)=\left(1+\frac{i^{(m)}}{m}\right)^{m} \tag{82}$$

Expanding equation (82), we obtain

$$\left(1+\frac{i^{(m)}}{m}\right)^{m}=1+m\times\left(\frac{i^{(m)}}{m}\right)^{1}+\frac{m(m-1)}{2!}\times\left(\frac{i^{(m)}}{m}\right)^{2}+\frac{m(m-1)(m-2)}{3!}\times\left(\frac{i^{(m)}}{m}\right)^{3}+\ldots \tag{83}$$

$$\left(1+\frac{i^{(m)}}{m}\right)^{m}$$
$$=1+m\times\left(\frac{i^{(m)}}{m}\right)^{1}+\frac{m(m-1)}{m^{2}}\times\frac{1}{2!}\left(i^{(m)}\right)^{2}+\frac{m(m-1)(m-2)}{m^{3}}\times\frac{1}{3!}\left(i^{(m)}\right)^{3}+\ldots \tag{84}$$

$$\lim_{m\to\infty}\left(1+\frac{i^{(m)}}{m}\right)^m$$
$$=\lim_{m\to\infty}\left\{\begin{matrix}1+m\times\left(\dfrac{i^{(m)}}{m}\right)^1+\dfrac{m(m-1)}{m^2}\times\dfrac{1}{2!}\left(i^{(m)}\right)^2+\dfrac{m(m-1)(m-2)}{m^3}\times\dfrac{1}{3!}\left(i^{(m)}\right)^3\\ +\ldots+\ldots\end{matrix}\right\} \tag{85}$$

$$\lim_{m\to\infty}\left(1+\frac{i^{(m)}}{m}\right)^m$$
$$=\lim_{m\to\infty}\left\{1+i^{(m)}+\frac{m^2\left(1-\frac{1}{m}\right)}{m^2}\times\frac{1}{2!}\left(i^{(m)}\right)^2+\frac{m^3\left(1-\frac{1}{m}\right)\left(1-\frac{2}{m}\right)}{m^3}\times\frac{1}{3!}\left(i^{(m)}\right)^3+\ldots\right\} \tag{86}$$

$$\lim_{m\to\infty}\left(1+\frac{i^{(m)}}{m}\right)^m=1+i^{(\infty)}+\frac{1}{2!}\left(i^{(\infty)}\right)^2+\frac{1}{3!}\left(i^{(\infty)}\right)^3+\frac{1}{4!}\left(i^{(\infty)}\right)^4\ldots \tag{87}$$

But by definition,

$$e^{\delta}=1+\frac{\delta}{1!}+\frac{\delta^2}{2!}+\frac{\delta^3}{3!}+\frac{\delta^4}{4!}\ldots \tag{88}$$

Consequently, comparing equations (87) and (88) yields

$$\lim_{m\to\infty} i^{(m)}=\delta \tag{89}$$

Consequently,

$$\lim_{m\to\infty}\left(1+\frac{i^{(m)}}{m}\right)^m=e^{\delta} \tag{90}$$

This implies that,

$$e^{-\delta}=\left(1-\frac{\delta}{m}\right)^m \tag{90a}$$

From the argument advanced above, the force of interest in equation (80) may possibly become intractable to execute for life insurance valuation since in general, the accumulation function does not have a closed form expression. Therefore, the deterministic market solves this problem and life insurance assumes that investment starts at time zero. This translates to setting $s=0$ in $A(0,t)$ so that $\delta(0,t)=\delta$ is constant, to enable the evaluation of the force of interest. Moreover, life insurers usually issue guaranteed fixed interest rates in dealing with survival benefits products, such as life annuity and defined benefits schemes, and as a result, the particular case where $\delta(0,t)=\delta$ for all t is of practical importance in life insurance valuation. Following the callistenics in Ogungbenle *et al.* (2024), the interest rate intensity $\delta_k(.)$ depending on i was modelled as

$$\left[\frac{k^2}{12}-\frac{1}{12}\left(\frac{k-\left(\frac{1-v^k}{i}\right)}{\left(\frac{1-v^k}{i}\right)}\right)-\frac{1}{12}\right]\times\ln\left|\left(\frac{1}{1-d}\right)\right|^2$$

$$=\left(-\left\{\frac{1}{2}\left(\frac{k-\left(\frac{1-v^k}{i}\right)}{\left(\frac{1-v^k}{i}\right)}\right)+\frac{k}{2}+\frac{1}{2}\right\}+\sqrt{\begin{array}{l}\left\{\frac{1}{2}\left(\frac{k-\left(\frac{1-v^k}{i}\right)}{\left(\frac{1-v^k}{i}\right)}\right)+\frac{k}{2}+\frac{1}{2}\right\}^2\\+4\left[\frac{k^2}{12}-\frac{1}{12}\left(\frac{k-\left(\frac{1-v^k}{i}\right)}{\left(\frac{1-v^k}{i}\right)}\right)-\frac{1}{12}\right]\left(\frac{k-\left(\frac{1-v^k}{i}\right)}{\left(\frac{1-v^k}{i}\right)}\right)\end{array}}\right) \tag{91}$$

where $\delta=\ln\left|\left(\frac{1}{1-d}\right)\right|$, k is the investment period, $\left(\frac{1-v^k}{i}\right)=a_{\overline{k|}}$ is the ordinary annuity and $\frac{1}{1-d}=\frac{1}{1-\frac{i}{1+i}}=1+i$. However, the authors did not investigate the behaviour of the intensities from 6% to 10% which actuaries use in generating life tables. For a small but finite time horizon $\Delta>0$, the error function $\in(\Delta)$ can be modelled as

$$\in(\Delta)=A(s,s+\Delta)-\exp\left(\int_u^{u+\Delta}\delta(\xi)d\xi\right) \tag{91a}$$

$$\Rightarrow\lim_{\Delta\to 0}\in(\Delta)=\lim_{\Delta\to 0}\left[A(s,s+\Delta)-\exp\left(\int_u^{u+\Delta}\delta(\xi)d\xi\right)\right] \tag{91b}$$

$$\Rightarrow\lim_{\Delta\to 0}\in(\Delta)=A(s,s)-\exp\left(\int_u^{u}\delta(\xi)d\xi\right) \tag{91c}$$

$$\Rightarrow\lim_{\Delta\to 0}\in(\Delta)=1-\exp(0)=1-1=0 \tag{91d}$$

where $\in(\Delta)=O(\Delta^2);\quad \Delta\to 0$ and $\in(\Delta)$ function evaluates performance away from the expansion point. The error vanishes as $\Delta\to 0$ and validates the Taylor-based modelling equations and the order of the error yields a transparent measure of estimation quality. If $\delta(\xi)=\delta$ constant, then

$$\in(\Delta)=A(s,s+\Delta)-\exp\left(\int_u^{u+\Delta}\delta d\xi\right) \tag{91e}$$

$$\Rightarrow\in(\Delta)=A(s,s+\Delta)-\exp(\delta\Delta) \tag{91f}$$

2.1 Presentation of Results

Table 1: Estimated value of interest rate intensity $\delta(i)$

k	$\delta(0,0.06)$	$\delta(0,0.07)$	$\delta(0,0.08)$	$\delta(0,0.09)$	$\delta(0,1.0)$
1	0.0582689091	0.0676586504	0.0769610449	0.0861777028	0.0953101907
2	0.0582687565	0.0676583752	0.0769605877	0.0861769894	0.0953091313
3	0.0582683103	0.0676575736	0.0769592611	0.0861749277	0.0953060813
4	0.0582674107	0.0676559637	0.0769566072	0.0861708183	0.0953000248
5	0.0582659101	0.0676532885	0.0769522135	0.0861640403	0.0952900715
6	0.0582636723	0.0676493139	0.0769457101	0.0861540441	0.0952754450
7	0.0582605715	0.0676438273	0.0769367656	0.0861403452	0.0952554716
8	0.0582564921	0.0676366358	0.0769250841	0.0861225186	0.0952295714
9	0.0582513279	0.0676275652	0.0769104034	0.0861001942	0.0951972497
10	0.0582449815	0.0676164585	0.0768924916	0.0860730525	0.0951580902
11	0.0582373636	0.0676031750	0.0768711456	0.0860408205	0.0951117482
12	0.0582283932	0.0675875894	0.0768461888	0.0860032686	0.0950579446
13	0.0582179964	0.0675695903	0.0768174696	0.0859602067	0.0949964610
14	0.0582061066	0.0675490801	0.0767848595	0.0859114821	0.0949271338
15	0.0581926637	0.0675259738	0.0767482514	0.0858569758	0.0948498507
16	0.0581776142	0.0675001981	0.0767075587	0.0857966007	0.0947645455
17	0.0581609104	0.0674716911	0.0766627134	0.0857302987	0.0946711946
18	0.0581425104	0.0674404014	0.0766136651	0.0856580388	0.0945698127
19	0.0581223776	0.0674062875	0.0765603797	0.0855798147	0.0944604500
20	0.0581004807	0.0673693172	0.0765028384	0.0854956425	0.0943431880
21	0.0580767933	0.0673294671	0.0764410364	0.0854055596	0.0942181367
22	0.0580512936	0.0672867220	0.0763749819	0.0853096221	0.0940854318
23	0.0580239640	0.0672410744	0.0763046954	0.0852079033	0.0939452313
24	0.0579947914	0.0671925240	0.0762302083	0.0851004922	0.0937977134
25	0.0579637663	0.0671410773	0.0761515623	0.0849874916	0.0936430735
26	0.0579308833	0.0670867471	0.0760688086	0.0848690170	0.0934815220
27	0.0578961402	0.0670295520	0.0759820068	0.0847451949	0.0933132822
28	0.0578595382	0.0669695161	0.0758912242	0.0846161616	0.0931385879
29	0.0578210818	0.0669066683	0.0757965352	0.0844820618	0.0929576818
30	0.0577807784	0.0668410426	0.0756980206	0.0843430476	0.0927708132

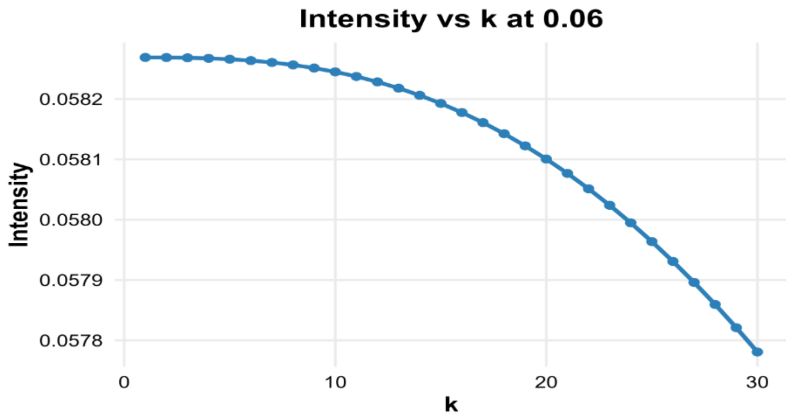

Figure 1

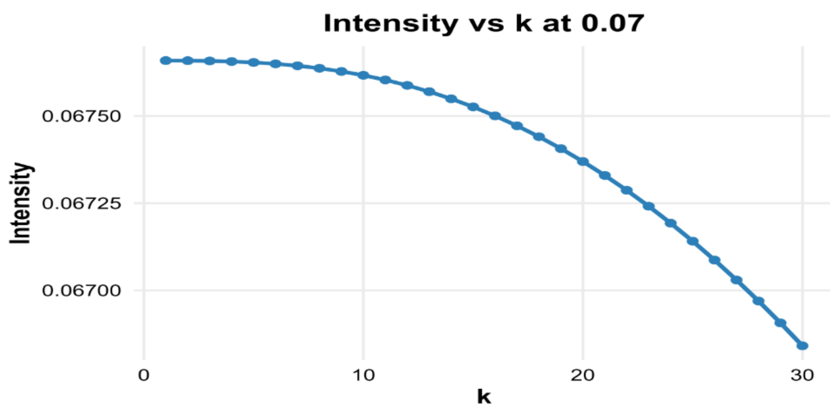

Figure 2

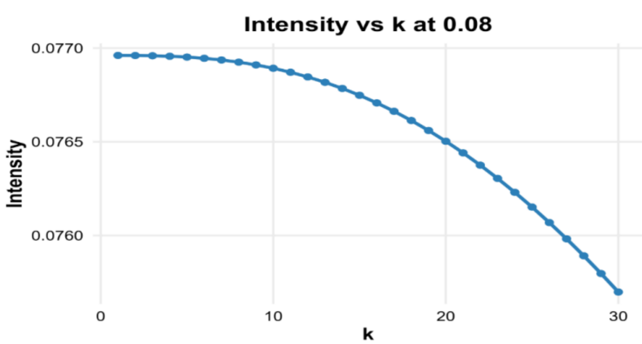

Figure 3

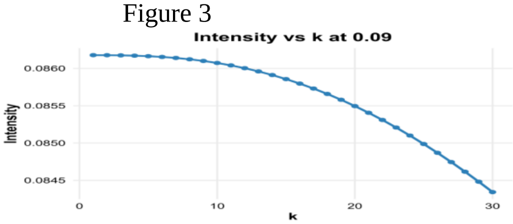

Figure 4

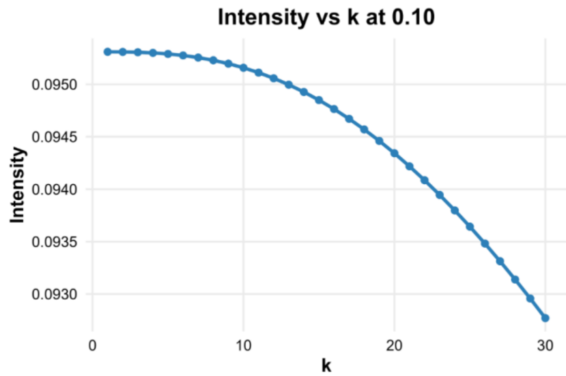

Figure 5

3.0 Discussion of Results

The previous arguments presented from equations (54) to (80) have some interesting implications. Suppose we let Δs and Δt be infinitesimally small times in the intervals $[s, s+\Delta s]$ and $[t, t+\Delta t]$ respectively. Since $A(s,t)$ is differentiable, the change in the accumulation function is

$$\Delta A = A(s+\Delta s, t+\Delta t) - A(s,t) \tag{92}$$

By definition

$$\frac{\partial}{\partial s} A(s, t+\Delta t) = \lim_{\Delta s \to 0} \left\{ \frac{A(s+\Delta s, t+\Delta t) - A(s, t+\Delta t)}{\Delta s} \right\} \tag{93}$$

But the change in accumulation function (92) is equal to

$$\Delta A = \delta(s,t) A(s,t) \Delta s \times \Delta t + h(\Delta s, \Delta t) \tag{94}$$

The actual change in accumulation function is given by $\Delta A \times \Delta s$ and (94) yields

$$\Delta A \times \Delta s = \delta(s,t) A(s,t) \Delta s \times \Delta t + h(\Delta s, \Delta t) \tag{95}$$

$h(\Delta s, \Delta t)$ is infinitesimally small. Consequently, the right-hand side of (94) becomes

$$A(s+\Delta s, t+\Delta t) \times \Delta s - A(s,t) \times \Delta s = \delta(s,t) A(s,t) \Delta s \times \Delta t + h(\Delta s, \Delta t) \tag{96}$$

where

$$\lim_{\substack{\Delta s \to 0 \\ \Delta t \to 0}} h(\Delta s, \Delta t) \to 0 \tag{97}$$

Now add and subtract $A(s, t+\Delta t)\Delta s$ from the left-hand side of (96), to obtain

$$\begin{aligned} &A(s+\Delta s, t+\Delta t) \times \Delta s - A(s, t+\Delta t)\Delta s + A(s, t+\Delta t)\Delta s - A(s,t)\Delta s \\ &= \delta(s,t) A(s,t) \Delta s \Delta t + h(\Delta s, \Delta t) \end{aligned} \tag{98}$$

Divide (98) through by $\Delta s \times \Delta t$ to obtain the partial derivatives

$$\begin{aligned} &\frac{A(s+\Delta s, t+\Delta t) - A(s, t+\Delta t) + A(s, t+\Delta t) - A(s,t)}{\Delta t} \\ &= \delta(s,t) A(s,t) + \frac{h(\Delta s, \Delta t)}{\Delta s \Delta t} \end{aligned} \tag{99}$$

Taking the limit of (99) as $\Delta t \to 0$ results to

$$\left\{\lim_{\Delta t\to 0}\frac{A(s+\Delta s,t+\Delta t)-A(s,t+\Delta t)}{\Delta t}+\lim_{\Delta t\to 0}\frac{A(s,t+\Delta t)-A(s,t)}{\Delta t}\right\}$$
$$=\lim_{\Delta t\to 0}\delta(s,t)A(s,t)+\lim_{\substack{\Delta t\to 0\\ \Delta s\to 0}}\frac{h(\Delta s,\Delta t)}{\Delta s\Delta t} \quad (100)$$

Consequently from (100), the interest rate intensity is obtained as

$$\frac{\partial A(s,t)}{\partial s}+\frac{\partial A(s,t)}{\partial t}=\delta(s,t)A(s,t) \quad (101)$$

$$\Rightarrow \delta(s,t)=\frac{1}{A(s,t)}\left\{\frac{\partial A(s,t)}{\partial s}+\frac{\partial A(s,t)}{\partial t}\right\} \quad (102)$$

$$\delta(s,t)=\frac{1}{A(s,t)}\left\{\frac{\partial}{\partial s}+\frac{\partial}{\partial t}\right\}A(s,t) \quad (103)$$

The multicurve function is formulated as

$$A_k(s+\Delta_1,s+\Delta_2)=A_k(s,s)+\frac{\partial A_k}{\partial s_1}\Delta_1+\frac{\partial A_k}{\partial s_2}\Delta_2+... \quad (103a)$$

Consequently, the vector of interest rates is

$$\delta(s)=\left(\delta_1(s),\delta_2(s),\delta_3(s),...,\delta_M(s)\right)^T \quad (103b)$$

If the error

$$\in(\Delta)=A(s,s+\Delta)-\exp\left(\int_s^{s+\Delta}\delta(\xi)d\xi\right) \quad (103c)$$

is small for different $\delta(s)$ inputs, it then signifies calibration stability.

Since $A(s,s+\Delta)$ in (103c) in Table 1 does not have a closed form expression, we alternatively specify the tolerance limit a priori $\varepsilon=0.5\%$ and the absolute value of the difference falls within the tolerance limit. That is, $|\delta_{est}-\delta_{exact}|<\varepsilon$ where $\varepsilon>0$ is a small number. The exact values of the force of interest at 6%, 7%, 8%, 9%, and 10% are:

$\ln(1.06)=0.0582689081$; $\ln(1.07)=0.0676586485$; $\ln(1.08)=0.0769610411$;

$\ln(1.09)=0.0861776962$ and $\ln(1.1)=0.0953101798$.

The Table 1 compares the force of interest over 30 years for an investment under various annual interest rates: 6%, 7%, 8%, 9%, and 10%. We confirm that

$$\delta_k(0,i)\xrightarrow{convergence}\log_e(1+i); \quad i=0.06,0.07,0.08,0.09,0.10 \quad ; \quad k=1-30. \quad (103d)$$

The force of interest, which reflects how continuously compounded interest accumulates over time, is shown for each year from year 1 to year 30. A key observation is that the force of interest decreases steadily over time for all interest rates. This gradual decline is expected in continuously compounded interest systems, where the growth slows down as the investment's value increases, reflecting the diminishing effect of compound interest as the period of investment lengthens. Across all interest rates, the force of interest decreases, though at different initial levels. For example, the force of interest at 6% begins at 0.0583 in year 1 and falls to 0.0578 by year 30, while the force of interest at 10% starts at 0.0953 in year 1 and decreases to 0.0928 by year 30. The rate of decrease is relatively consistent across all interest rates, with the decline slowing down over time. This indicates that although higher interest rates start with larger forces

of interest, the rate at which they decrease over time is similar to lower interest rates.
The behavior of the force of interest in Table 1 and Figures 1,2,3,4, and 5 reveal that the largest declines occur early in the investment period. The trajectories in Figures 1,2,3,4,5 are monotone decreasing. This non-linear decrease is typical for continuously compounded interest, where the initial effect of compounding is more pronounced and then gradually levels off as the value of the investment increases. For example, the force of interest at 10% decreases by approximately 0.0025 over 30 years, while the force of interest at 6% shows a similar absolute decrease. The consistent gap between the rates (around 0.009 between each consecutive interest rate) reflects how the force of interest correlates directly with the nominal interest rate, maintaining a proportional relationship across the different interest levels. From a financial perspective, the diminishing force of interest implies that higher interest rates, while yielding greater returns early in the investment, lose their advantage over time. In contrast, lower interest rates, though starting with smaller forces of interest, exhibit more stable growth throughout the period. This gradual decline in force of interest suggests that the choice of an interest rate has a diminishing impact as time progresses, making long-term financial strategies less dependent on the initial rate and more focused on other factors like investment duration and compounding frequency. The table also provides valuable insights for investors planning long-term investments. While starting with a higher interest rate (e.g., 9% or 10%) may seem advantageous initially, the benefit of these higher rates diminishes over time as the force of interest decreases at a similar rate for each interest level. Therefore, understanding the time-sensitive nature of compound interest and how the force of interest decreases over an extended period can help investors make more informed decisions about which interest rates to target for long-term growth. In conclusion, the table highlights the crucial interplay between interest rates and the force of interest over time. While higher interest rates offer greater initial returns, the diminishing effect of continuous compounding over long periods makes the differences less pronounced as time progresses. For long-term investors, it's important to consider not only the initial interest rate but also how the force of interest will behave throughout the investment period, as the impact of compounding reduces with time.

4.0 Conclusion

In conclusion, the table illustrates how the force of interest behaves under different interest rates over a 30-year investment horizon. While higher interest rates initially offer larger forces of interest, the decline in the force of interest over time is relatively consistent across all rates. This gradual decrease reflects the diminishing effect of compound interest as the investment grows, highlighting that the impact of the interest rate lessens as time progresses. Investors should be aware that while higher interest rates provide more substantial initial returns, the difference between rates diminishes over time, making the long-term benefits of a higher interest rate less significant. Therefore, when planning for long-term investments, it is essential to consider not just the starting interest rate, but also how the force of interest evolves throughout the investment period. The results suggest that, over extended periods, the choice of interest rate should be balanced with other factors such as investment duration, compounding frequency, and overall investment strategy. Future research trend may focus on the implications of the force of interest on annuities.

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