Three-Step Third-Derivative Block Multistep Algorithm for Solving Second-Order Nonlinear Lane–Emden-Type Differential Equations

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Abstract
In this paper, a direct solution for some well-known classes of Lane–Emden-type second-order nonlinear ordinary differential equations is proposed, without converting them into a first-order system of equations by using a new class of third-derivative block multistep methods. These methods were derived from a continuous scheme through an interpolation and collocation technique and are assembled in block forms to produce the numerical solution in the specified interval on the entire range of integration. The properties of the block method are discussed and the efficiency of the method is shown when applied on some second-order nonlinear Lane–Emden-type differential equations. It was observed that the method was consistent, zero stable, convergent and is stable in the interval [–4.4, 0]. The result shows that the method is suitable for the solution of the nonlinear Lane–Emden-type equations and performs better when compared to those in the Literature.

Keywords: block multistep algorithm, continuous schemes, differential equation, Lane–Emden

Introduction

Consider the general second order equation

$$y'' = f(x, y, y'), \quad y(x_0) = y_0, \quad y'(x_0) = y'_0$$
$$x \in [x_0, \ x_N] \qquad (1)$$

where $f\colon \Re \times \Re^{2m} \to \Re^m$ are continuous functions, and m is the dimension of the system.

The second order nonlinear Lane–Emden-type equation is generally formulated as

$$y'' + \frac{\tau}{x} y' + p(x) r(y) = q(x),$$
$$x > 0, \ \tau > 0 \quad y(0) = \alpha \ , \quad y'(0) = \beta \qquad (2)$$

where α, β are real constants $p(x)$, $r(y)$ and $q(x)$ are some given functions. For other special forms of $r(y)$, the well-known Lane–Emden equations are used to model several phenomena in mathematical physics and astrophysics such as the theory of stellar structure, the thermal behaviour of a spherical cloud of gas, isothermal gas spheres and the theory of thermionic currents (Hojjati and Parand, 2011).

If $\tau = 2, \ q(x) = 0$ then equation (1) becomes

$$y'' + \frac{2}{x} y' + p(x) r(y) = 0,$$
$$y(0) = \alpha \ , \quad y'(0) = \beta \qquad (3)$$

Equation (3) is a special case of (1) and is called the generalised Emden–Fowler differential equation (Parand and Shanini, 2010). Several second-order non-linear ordinary differential equations of Lane–Emden-type are derived as special cases of (3). Examples of such are: when $r(y) = y^m$, $p(x) = 1$ then (3) becomes

$$y'' + \frac{2}{x} y' + y^m = 0 \ , \ y(0) = 0 \ , \quad y'(0) = 1 \qquad (4)$$

According to Davis (1962) and Shawagfeh (1993), equation (4) is referred to as the standard Lane–Emden equation, which is used in the modelling of temperature variations of a spherical cloud gas under the mutual attraction of its molecules subject to the law of classical thermodynamics.

Also, if $r(y) = (y^2 - C)^{\frac{3}{2}}$, $p(x) = 1$ equation (3) yields $y(0) = 1$, $y'(0) = 0$ (5)

Equation (5) is known as the white dwarf equation, introduced by Davis (1962) in the study of the gravitational potential of the degenerate white dwarf stars. In recent times, analytical solutions have been proffered for the solution of equation (1) (He, 2003; Liao, 2003) but the main difficulty arises at the point $x = 0$, called singular point. This has prompted several researchers to propose other techniques, which are based on either series solutions or perturbation techniques. These include Shawagfeh (1993), Wazwaz (2001), Mandelzweig and Tabakin (2001), He (2003), Horedt (2004), Yousefi (2006), Ramos (2008), Parand *et al.* (2009) and Shiralashetti *et al.* (2015). Despite the successes recorded by earlier researchers on the Lane–Emden-type equation (2), most of the methods used are analytical or semianalytical-based. Numerical methods such as the second derivative multistep method by Hojjati and Parand (2011) have been proposed for the solution of the Lane–Emden-type problems using the collocation and interpolation approach. These methods are implemented by first reducing the equation (1) to a first-order system of equations.

Thus, in this paper, we propose a third derivative block algorithm (TDBA) through the collocation and interpolation technique as in the block Nyström method (Jator and Oladejo, 2017; Okunuga *et al*., 2012) since the technique produces a continuous scheme, which is used to derive the complementary and the main method that shall be used together as a single block algorithm for the direct solution of the Lane–Emden-type problems (1) without reducing it to system of first-order equations.

To this end, a new third-derivative block method has been developed, analysed and implemented as a self-starting method on some well-known classes of Lane–Emden-type second-order nonlinear ordinary differential equations, without converting them into first-order system of equations.

Materials and Methods

Derivation

In order to develop the third-derivative block algorithm (TDBA), the interval $x_n \le x \le x_{n+4}$ is considered. Assuming that the exact solution to (4) is approximated by a power series of the form,

$$v(x) = \sum_{j=0}^{7} b_j x^j \tag{6}$$

where b_0, b_1,…, b_7, are coefficients to be uniquely determined. In order to determine these coefficients, the following eight conditions must be satisfied.

$$\left.\begin{aligned} v(x_n) &= y_{n+i}, & i &= 0, 1. \\ v''(x_n) &= f_{n+i}, & i &= 1, 2, 3 \\ v'''(x_n) &= g_{n+i}, & i &= 1, 2, 3, \end{aligned}\right\} \tag{7}$$

By differentiating (6) twice and thrice, we obtain, respectively:

$$v''(x) = \sum_{j=2}^{7} j(j-1)b_j x^{j-2} = f(x, y, y')$$

$$v'''(x) = \sum_{j=2}^{7} j(j-1)(j-2)b_j x^{j-3} = \frac{d(f(x, y, y'))}{dx} \tag{8}$$

By using (6) and (8) in the conditions given in (7) we obtain a system of nonlinear equations of the form, which is solved using the Gaussian elimination method, enhanced by Maple 17, to obtain the unknown coefficients $b_j,\ j = 0,1, \dots 7$.

$$\begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 & x_{n+1}^7 \\ 0 & 0 & 2 & 6x_{n+1} & 12x_{n+1}^2 & 20x_{n+1}^3 & 30x_{n+1}^4 & 42x_{n+1}^5 \\ 0 & 0 & 2 & 6x_{n+2} & 12x_{n+2}^2 & 20x_{n+2}^3 & 30x_{n+2}^4 & 42x_{n+2}^5 \\ 0 & 0 & 2 & 6x_{n+3} & 12x_{n+3}^2 & 20x_{n+3}^3 & 30x_{n+3}^4 & 42x_{n+3}^5 \\ 0 & 0 & 0 & 6 & 24x_{n+1} & 60x_{n+1}^2 & 120x_{n+1}^3 & 210x_{n+1}^4 \\ 0 & 0 & 0 & 6 & 24x_{n+2} & 60x_{n+2}^2 & 120x_{n+2}^3 & 210x_{n+2}^4 \\ 0 & 0 & 0 & 6 & 24x_{n+3} & 60x_{n+3}^2 & 120x_{n+3}^3 & 210x_{n+3}^4 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \\ b_7 \end{bmatrix} = \begin{bmatrix} y_n \\ y_{n+1} \\ f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ g_{n+1} \\ g_{n+2} \\ g_{n+3} \end{bmatrix} \tag{9}$$

These coefficients are then substituted into (6) to get the continuous form of the Method given as

$$v(x) = \sum_{j=0}^{1} \alpha_j(x) y_{n+j} + h^2 \sum_{j=1}^{3} \beta_j(x) f_{n+j} + h^3 \sum_{j=1}^{3} \gamma_j(x) g_{n+j} \tag{10}$$

where $h = x_{n+1} - x_n$.

Differentiating (10) we obtain the first derivative as $v'(x) = \frac{d}{dx} v(x)$ (11)

We assume that y_n is the numerical approximation to the analytical solution $v(x_n)$, y'_n, is the numerical approximation to f_{n+jh} is the numerical approximation to $f(x_{n+jh}, y_{n+jh}, y'_{n+jh})$.

In order to obtain the $v'(x_n)$ main methods, equation (10) was evaluated to give the following:

$$y_{n+i} = -(i-1)y_n + i\, y_{n+1} + h^2 \sum_{j=1}^{3} \beta_j(i) f_{n+j} + h^3 \sum_{j=1}^{3} \gamma_j(i) g_{n+j}, \quad i = 2,3 \tag{12}$$

The additional methods are obtained by evaluating (11) to give the following:

$$hy'_{n+i} = -y_n + y_{n+1} + h^2 \sum_{j=1}^{3} \overline{\beta_j(i)} f_{n+j} + h^3 \sum_{j=1}^{3} \overline{\gamma_j(i)} g_{n+j} \qquad , i = 0, 1, 2, 3 \tag{13}$$

The discrete forms of equations (12) and (13) are given as follows

$$\left.\begin{aligned} y_{n+2} &= -y_n + 2y_{n+1} - \frac{29}{120}h^2 f_{n+1} + \frac{11}{15}h^2 f_{n+2} + \frac{61}{120}h^2 f_{n+3} - \frac{3}{5}h^3 g_{n+1} - h^3 g_{n+2} - \frac{3}{20}h^3 g_{n+3} \\ y_{n+3} &= -2y_n + 3y_{n+1} - \frac{7}{20}h^2 f_{n+1} + \frac{11}{5}h^2 f_{n+2} + \frac{23}{20}h^2 f_{n+3} - \frac{47}{40}h^3 g_{n+1} - 2h^3 g_{n+2} - \frac{13}{40}h^3 g_{n+3} \end{aligned}\right\} \tag{14}$$

$$\left.\begin{aligned} hy'_n &= -y_n + y_{n+1} + \frac{5717}{1680}h^2 f_{n+1} - \frac{59}{30}h^2 f_{n+2} - \frac{3253}{1680}h^2 f_{n+3} + \frac{212}{105}h^3 g_{n+1} + \frac{751}{210}h^3 g_{n+2} + \frac{97}{180}h^3 g_{n+3} \\ hy'_{n+1} &= -y_n + y_{n+1} - \frac{463}{840}h^2 f_{n+1} + \frac{17}{30}h^2 f_{n+2} + \frac{407}{840}h^2 f_{n+3} - \frac{1067}{1680}h^3 g_{n+1} - \frac{97}{105}h^3 g_{n+2} - \frac{241}{1680}h^3 g_{n+3} \\ hy'_{n+2} &= -y_n + y_{n+1} - \frac{73}{560}h^2 f_{n+1} + \frac{11}{10}h^2 f_{n+2} + \frac{297}{560}h^2 f_{n+3} - \frac{61}{105}h^3 g_{n+1} - \frac{229}{210}h^3 g_{n+2} - \frac{131}{840}h^3 g_{n+3} \\ hy'_{n+3} &= -y_n + y_{n+1} - \frac{71}{840}h^2 f_{n+1} + \frac{49}{30}h^2 f_{n+2} + \frac{799}{840}h^2 f_{n+3} - \frac{191}{336}h^3 g_{n+1} - \frac{97}{105}h^3 g_{n+2} - \frac{353}{1680}h^3 g_{n+3} \end{aligned}\right\} \tag{15}$$

Convergence Analysis

Basic Definitions

The methods generated by (12), (14) and (15) are combined to give the TDBA which is conveniently analyzed and implemented in a block-by-block fashion by defining the vectors $Y_{\varpi}, Y_{\varpi-1}, F_{\varpi}, F_{\varpi-1},$ as follows:

$$Y_{\varpi} = \left(y_{n+1}, y_{n+2}, y_{n+3}, hy'_{n+1}, hy'_{n+2}, hy'_{n+3}\right)^T$$

$$F_{\varpi} = \left(f_{n+1}, f_{n+2}, f_{n+3}, h\, g_{n+1}, h\, g_{n+2}, hg_{n+3}\right)^T$$

$$Y_{\varpi-1} = \left(y_{n-3}, y_{n-2}, y_{n-1}, y_n, hy'_{n-3}, hy_{n-2}, hy_{n-1}, hy_n\right)^T$$

Thus, the TDBA is given by

$$A^{(1)}Y_{\varpi} = A^{(0)}Y_{\varpi-1} + h^2 B^{(1)}F_{\varpi} \tag{16}$$

where $\varpi = 1, \cdots, N$, $n = 0, 1, \cdots, N-3$, $A^{(r)}$, $r = 0, 1$, $B^{(1)}$ are 8 by 8 matrices whose entries are given by the coefficients of (14) and (15)

$$A^{(1)} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 & 0 \\ -3 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad A^{(0)} = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix}$$

$$B^{(1)} = \begin{bmatrix} \frac{5717}{1680} & -\frac{59}{30} & -\frac{3253}{1680} & \frac{212}{105} & \frac{751}{210} & \frac{97}{168} \\ -\frac{29}{120} & \frac{11}{15} & \frac{61}{120} & -\frac{3}{5} & -1 & -\frac{3}{20} \\ -\frac{7}{20} & \frac{11}{5} & \frac{23}{20} & -\frac{47}{40} & -2 & -\frac{13}{40} \\ -\frac{463}{840} & \frac{17}{30} & \frac{407}{840} & -\frac{1067}{1680} & -\frac{97}{105} & -\frac{241}{1680} \\ -\frac{73}{560} & \frac{11}{10} & \frac{297}{560} & -\frac{61}{105} & -\frac{229}{210} & -\frac{131}{840} \\ -\frac{71}{840} & \frac{49}{30} & \frac{799}{840} & -\frac{191}{336} & -\frac{97}{105} & -\frac{353}{1680} \end{bmatrix}$$

Definition 1. The TDBA (16) is said to be consistent if it has order $p \geq 1$.

Definition 2. The TDBA (16) is zero stable provided the roots R_j, j = 1,…, 2k of its first characteristic polynomial satisfy $|R_j| \leq 1$, j = 1,…, 2k and for those roots with $|R_j| = 1$, the multiplicity does not exceed 2 (Fatunla, 1991).

Local Truncation Error

The local truncation error associated with the third-derivative block algorithm can be represented by the linear difference operator

$$L[y(x);h] = \sum_{j=0}^{k} \alpha_j y(x_n + jh) - h^2\beta_j y''(x_n + jh) - h^3\gamma_j y'''(x_n + jh)$$

Assuming that $y(x)$ is sufficiently differentiable, the terms in (16) can be expanded as a Taylor series about the point x_n to obtain the expressions for the LTEs as:

$$L[y(x) : h] = \\ = C_0 y(x) + C_1 h y'(x) + C_2 h^2 y''(x) + . \ . \ . + C_q h^q y^q(x) + \cdots$$

where C_i, $i = 0,1,2,\ldots$ are constant coefficients.

The TDBA is said to have algebraic order of accuracy q if

$$C_0 = C_1 = C_2 \ . \ . \ . C_q = 0 \ , \quad C_{q+1} = 0, \quad C_{q+2} \neq 0$$

such that the local truncation error

$$\|E_\varpi\| = C_{q+2}h^{q+2} + O\left(h^{q+3}\right) \qquad (17)$$

where $\|.\|$ is the maximum norm. Therefore, C_{q+2} is the local truncation error of the method given by the vector $\left(y_{n+1}, y_{n+2}, y_{n+3}, \ hy'_{n+1}, \ hy'_{n+2}, hy'_{n+3}\right)^T$ which is a member of the block method (15) and is given, respectively, by

$$C_8 = \left(\frac{5501}{604800}, \frac{659}{302400}, \frac{449}{100800}, \frac{1283}{604800}, \frac{449}{201600}, \frac{1411}{604800}\right)^T$$

with order $q = (6, 6, 6, 6, 6, 6)^T$.

Zero Stability

The zero stability of the TDBA is determined as the limit h tends to zero. Thus as $h \to 0$ the method (16) tends to the difference system

$$A^{(1)}Y_\varpi = A^{(0)}Y_{\varpi-1}$$

which is normalised to obtain the first characteristic polynomial $\rho(R)$ given by

$$\rho(R) = \det\left(RA^{(1)} - A^{(0)}\right) = -R^4(R-1)^2$$

The block method (16) is zero stable for $\rho(R) = 0$ and satisfied $|R_j| \leq 1, j = 1, \ldots, 2k$ and for those roots with $|R_j| = 1$, the multiplicity is simple and it does not exceed 2. Therefore the block method is zero-stable.

Linear Stability

The linear stability property of the TDBM is discussed by applying (16) to the scalar test equation

$$y'' = \lambda y$$

where λ is supposed to run through the negative eigenvalues of the Jacobian matrix $\frac{\partial f}{\partial y}$ (Sommeijer, 1993).

Letting z $= \lambda h^2$, it is shown that the application of (16) to the test equation yields $y_{\varpi+1} = M(z)y_\varpi$,

$$M(z) = \left(A^{(1)} - zB^{(1)}\right)^{-1}\left(A^{(0)}\right) \qquad (18)$$

where $M(z)$ is the amplification matrix that determines the stability of the method and its eigenvalues are the amplification factors.

Definition 4. The interval $[-\beta_0, 0]$ is the stability interval, if in this interval $\rho(M(z)) \leq 1$, where $\rho(M(z))$ is the spectral radius of $M(z)$ and β_0 is the stability boundary (see Sommeijer, 1993). Calculating the zeros of the polynomials $R_j(z)$, it is easily deduced that $R_j(z) \geq 0$, $j = 1, 2, 3$ $z \in [-4.4, 0]$, yielding $\beta_0 = 4.4$.

Numerical Examples

The TDBA was tested on some second-order nonlinear singular initial value problems of Lane–Emden-type differential equations to illustrate its accuracy and efficiency. A constant step-size is used in all the numerical examples. All computations are carried out using written codes in Maple 8.0. The maximum absolute error of the approximate solution on $[x_0, x_N]$ is calculated as

$$Error = Max|y(x) - y|$$

The rate of convergence (ROC) is calculated using the formula

$$ROC = log_2\left(\frac{Error^{2h}}{Error^h}\right) \qquad (19)$$

where $Error^h$ is the error obtained using the step-size h.

Example 1. The white dwarf equation is considered, introduced in Davis (1962) in the study of gravitational potential of the degenerate white dwarf stars.

$$y''(x) + \frac{2}{x}y'(x) + (y^2 - C)^{\frac{3}{2}} = 0, \ y(0) = 1, \ y'(0) = 0$$

It is solved with the TDBA for C = 0.2, 0.4, 0.6 and 0.8, where at C = 0. The example 1 reduces to the standard Lane–Emden equation of index *m* = 3.

Example 2. Consider the standard Lane–Emden equation that was used to model the thermal behaviour of a spherical cloud of gas acting under the mutual attraction of its molecules and subject to the classical laws of thermodynamics.

$$y''(x) + \frac{2}{x}y'(x) + y^m(x) = 0,$$
$$y(0) = 1, \quad y'(0) = 0, \quad x \geq 0$$

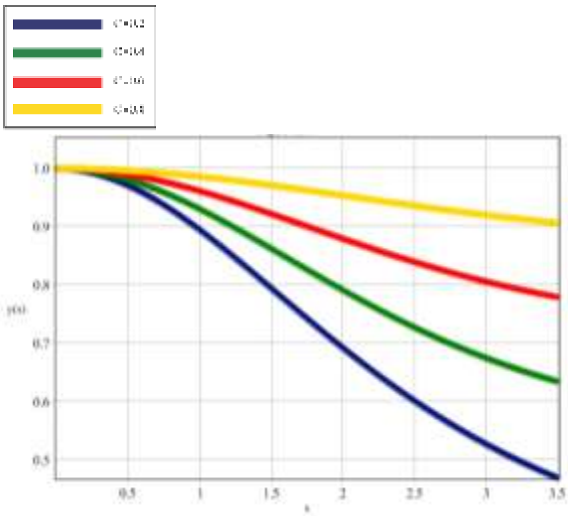

Figure 1: Numerical Result using the TDBM (Example 1)

Table 1: Results with *h* = 0.01 and m = 3, using TDBA (Example 2)

X	TDBA	Hojjati and Parand (2011)	Horedt (2004)
0.50	0.959839069944850	0.959839069883	0.959839
1.00	0.855057568588628	0.855057568546	0.855058
5.00	0.110819835139621	0.110819835160	0.110820
6.00	0.043737983889702	0.043737983910	0.043738

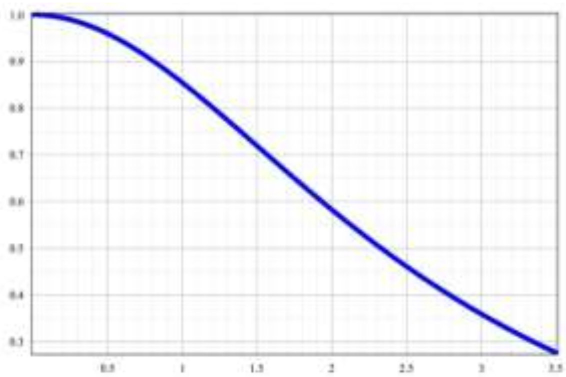

Figure 2: Numerical Result using TDBA (Example 2)

Example 3. Consider the nonlinear singular initial value problem, which was also solved by Shiralashetti *et al.* (2015) using Haar Wavelet Collocation Method,

$$y''(x) + \frac{6}{x}y'(x) + 14y(x) + 4y(x)log\big(y(x)\big) = 0,$$
$$y(0) = 1, \quad y'(0) = 0$$

where the analytic solution is given as

$$y(x) = e^{-x^2}, \quad x \in [0,1]$$

Table 2: Comparison of Errors (Example 3)

N	TDBA	ROC	HWCM
4	1.901×10^{-7}		–
8	4.211×10^{-7}	1.15	1.101×10^{-3}
16	8.183×10^{-9}	5.69	2.739×10^{-4}
32	1.401×10^{-10}	5.87	6.841×10^{-5}
64	2.226×10^{-12}	5.98	1.907×10^{-5}
128	3.491×10^{-14}	5.99	4.275×10^{-6}

Example 4. Consider the nonlinear second-order singular IVP that was solved using BVM3, BVM4 by Okunuga *et al.* (2012), and Shiralashetti *et al.* (2015) using Haar Wavelet Collocation Method (HWCM).

$$y'' + \frac{2}{x}y' + 4\left(2e^y + e^{\frac{y}{2}}\right) = 0, \quad y(0) = 0,$$
$$y'(0) = 0, \quad x \in [0,1]$$

where the analytic solution is given as

$$y(x) = -2\,ln(1 + x^2)$$

Table 3: Results with *h* = 0.01 (Example 4)

X	TDBA p=6	BVM3 p=4	BVM4 p=5
0.25	2.73×10^{-13}	9.35×10^{-8}	3.04×10^{-10}
0.50	5.69×10^{-13}	2.02×10^{-8}	3.09×10^{-10}
0.75	1.51×10^{-12}	9.72×10^{-9}	5.39×10^{-10}
1.00	1.50×10^{-12}	4.24×10^{-9}	3.34×10^{-10}

Table 4: Comparison of Errors (Example 4)

h	TDBA	HWCM
0.25	2.73×10^{-4}	–
0.125	2.272×10^{-6}	1.837×10^{-3}
0.0625	6.692×10^{-8}	4.701×10^{-4}
0.03125	1.385×10^{-10}	1.195×10^{-4}
0.015625	2.301×10^{-11}	3.010×10^{-5}
0.0078125	3.376×10^{-13}	7.555×10^{-6}

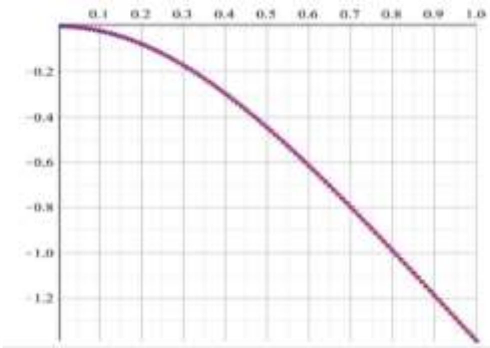

Figure 3: Numerical Result using TDBM (Problem 3.4)

Example 5. Consider the homogeneous singular IVP of the Lane–Emden-type equation, which has been solved by Shiralashetti *et al.* (2015) using HWCM,

$$y''(x) + \frac{2}{x}y'(x) = (4x^2 + 6)\,y(x), \quad y(0) = 1,$$
$$y'(0) = 0, \quad x \in [0,1]$$

where the analytic solution is given as

$$Exact: \quad y(x) = e^{x^2}$$

Table 5: Comparison of Errors (Example 5)

N	TDBM	ROC	HWCM
8	9.276×10^{-5}		3.723×10^{-4}
16	9.240×10^{-7}	6.65	1.186×10^{-4}
32	1.390×10^{-8}	6.05	3.141×10^{-5}
64	1.597×10^{-10}	6.44	7.907×10^{-6}
128	2.373×10^{-12}	6.07	1.998×10^{-6}

Results and Discussion

Figure 1 depicts the numerical solution trajectory using TDBA for different values of C at m = 3 for Example 1. This trajectory showed that it was in conformity with that of Davis (1962). In Table 1, the numerical results showed the superiority of the results obtained using TDBA over those obtained by Hojjati and Parand (2011) and Horedt (2004). The solution trajectory for Example 2 is shown in Figure 2. Table 2 shows the comparison of HWCM by Shiralashetti *et al.* (2015) and TDBA. It is seen that TDBA is more accurate with smaller errors at all points considered.

Figure 3 represents the solution trajectory to the exact solution. It shows a comparison of the TDBA to the method by Okunuga *et al.* (2012) for different end points. It is obvious, from Table 3, that our method is more accurate since the order is higher than that of Okunuga *et al.* (2012). Table 4 summarises the results obtained for Example 4 for different step lengths h, the maximum errors were compared with those of Shiralashetti *et al.* (2015). The TDBA was superior with a larger h. Table 5 shows that our method is consistent with the order of the method given by ROC and for different end points. The TDBA is more accurate than those reported by Shiralashetti *et al.* (2015).

Conclusion

In this paper, a 3-step third-derivative block algorithm (TDBA) for the direct solution of the Lane–Emden-type of second-order nonlinear initial value problems with singularity at x = 0 have been considered. The TDBA was applied directly to the Lane–Emden-type ordinary differential equations without reducing the differential equation to systems of first-order equation. The method was implemented in a block-by-block fashion and so does not suffer the disadvantages of requiring starting values and predictors, which are inherent in predictor–corrector methods. In general, it is shown that the computed ROC is higher but consistent with the theoretical order 6 of the TDBA. Numerical examples performed using TDBA show that the method is accurate and efficient.

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