



A bi-annual journal published by the Faculty of Science, University of Lagos, Nigeria

<http://jsrd.unilag.edu.ng/index.php/jsrd>

## A Generalized Henstock-type Integral for Gronwall-Bellman's Type Inequality

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(Received 29 December 2025/Revised 25 April 2026/Accepted 12 May 2026)

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### Abstract

In this paper, we establish some results involving the Generalized Henstock-type integral which generalizes Gronwall-Bellman's Type Inequality and discuss its applicability to some stochastic differential equations.

**MSC 2020:** 28B15, 46G10

**Keywords:** Henstock integral, Sequential Henstock integrals, Continuous functions, gauge functions.

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# 1 Introduction and Preliminaries

Henstock integral, introduced in the mid-1950s is a useful generalization of the Riemann integral. Henstock integral uses the Riemann sums and  $\epsilon - \delta$  definition to solve nowhere-continuous and extreme oscillatory functions. In 2016, the Sequential Henstock integral which involves the use of sequence of gauge functions was introduced (Paxton, 2016, Iluebe et. al. 2020) In the last ten years, a number of integration results for some generalized Riemann-type integrals for a certain class of real valued functions have been studied by many

researchers using different techniques, see for example, Mogbademu, Iluebe (2022) and the references therein.

In 1981, T. Gronwall obtained the Gronwall-Bellman inequality, which is an important tool in the study of qualitative behaviour of solutions of ordinary differential equations and integral equations. In 2013, Ostaszewski, K. Sochacki, presented a new generalization of the Gronwall-Bellman-type inequality and established some qualitative properties of nonlinear generalized Gronwall-Bellman-type integrals along with some applications to a new class of time-varying perturbed system.

**Lemma 1.1** (Gronwall Inequality) If  $F : [a,b] \times [a,b] \rightarrow \mathbb{R}$ ,  $s, t > 0$ , is a continuous function and there exist  $\xi$  and  $k \geq 0$  such that

$$F(t) \leq \xi + k \int_a^t F(s) ds \quad \text{for } s, t \in [a, b] \tag{1}$$

then

$$F(t) \leq \xi \exp(kt) \quad \text{for } s, t \in [a, b] \tag{2}$$

**Remark 1.2** Lemma 1.1 is the generalized version of Lemma 1.1 of (Gronwall, 1919).

**Lemma 1.3** (Gronwall-Bellman Inequality) Let  $F(t)$  and  $H(t)$  be nonnegative continuous functions on  $a \leq t \leq b$ , for which the inequality

$$F(t) \leq \xi + \int_a^t H(s)F(s)ds \quad \text{for } s,t \in [a,b] \tag{3}$$

holds, where  $\xi \geq 0$  is a constant. Then

$$F(t) \leq \xi \exp\left(\int_a^t H(s)ds\right) \quad \text{for } s,t \in [a,b] \tag{4}$$

**Remark 1.4** Lemma 1.3 is the generalized version of Lemma 1.2 of (Gronwall, 1919).

**Theorem 1.5** (Gronwall Inequality under the Henstock Integral) Let  $f : [a,b] \rightarrow \mathbb{R}$  be Henstock Kurzweil integrable. If there are constants  $\xi$  and  $k \geq 0$  such that

$$f(t) \leq \xi + k \int_a^t f(s)ds \quad \text{for } s,t \in [a,b] \tag{5}$$

then

$$f(t) \leq c \exp(kt) \quad \text{for } t \in [a,b] \tag{6}$$

if (5) holds a.e., then so does (6).

**Remark 1.6** Theorem 1.5 indicates that the Gronwall inequality under the Henstock integral is valid. Lemma 1.1 simplifies Lemma 1.3 and its generalizations are very useful in the qualitative analysis of the solutions to differential and integral equations; see (Gronwall, 1919, Bellman, 1953, Ding and Ye, 2009).

In literature, Ding and Ye (2009) considered some integral inequalities of Gronwall-Bellman’s type for generalized differential equation.

$$\frac{dx}{d\tau} = DF(x, t) \tag{7}$$

In 2016, Paxton investigated the sequential characterization of the Henstock integral for real valued functions, and then introduced some fundamental concepts useful in proving the properties of this integral. For example see Iluebe and Mogbademu (2022). Recently, Iluebe and Mogbademu (2024) established the generalized sequential

characterization for a certain class of Henstock-type integrals and proved some properties of these integrals in real spaces. For details of these results, the reader is referred to Iluebe and Mogbademu (2024) and the references therein. In this paper, we establish the Gronwall-Bellman inequality involving the Generalized Sequential Henstock integral that generalizes Gronwall-Bellman inequality, which enables us to study the stochastic differential equations and give a simple application of our results.

We first recall the following definitions whose results appeared in Iluebe and Mogbademu (2023).

**Definition 1.7** Let  $g : [a, b] \rightarrow \mathbb{R}$  be a function of bounded variation on  $[a, b]$ . A function  $F : [a, b] \times [a, b] \rightarrow \mathbb{R}$  is said to be Generalized Sequential Henstock Stieltjes integrable ( $GSH[a, b]$ ) to a number  $\alpha \in \mathbb{R}$  with respect to  $\nu$  on  $[a, b]$  if there exists a sequence of positive gauge functions  $\{\delta_n(t)\}_{n=1}^\infty$  on  $[a, b]$  such that for every  $\delta_n(t)$ -fine tagged partitions  $P_n = \{(u_{(i-1)_n}, u_{i_n}), t_{i_n}\}_{i=1}^{m_n}$ , we have

$$\sum_{n=1}^{m_n \in N} \{F(g(t_{i_n}, u_{i_n})) - F(g(t_{i_n}, u_{(i-1)_n}))\} \rightarrow \alpha \text{ as } n \rightarrow \infty.$$

We say  $\alpha = (GHS) \int_{[a,b]} F dv(t)$ . The set of all functions  $F$  which are

generalized Sequential Henstock Stieltjes integrable on  $[a, b]$  is denoted by  $GSHS[a, b]$ . If a subset  $E \subseteq [a, b]$  is given, then we write  $(GSHS) \int_E F =$

$(GSHS) \int X_E = F(E)$  for the Generalized Sequential Henstock integral

Stieltjes of  $F$  is the primitive of  $f$ .

**Remark 1.8** A special case of Definition 1.7 is discussed as follows: (i) If  $g(t)$  is an identity function of bounded variation on  $[a, b]$ , we obtain a definition for Generalized Sequential Henstock integral.

(ii) Setting  $F(t_{i_n}, u_{i_n}) = f(t_{i_n})u_{i_n}$  where  $f : [a, b] \rightarrow \mathbb{R}$  and  $t_{i_n}, u_{i_n} \in [a, b]$  with  $\delta_n(t) > 0$ , the Sequential Henstock integral is obtained (Iluebe and Mogbademu, 2023).

(iii) Considering  $F(t_i, u_i) = f(t_i)u_i$  where  $f : [a, b] \rightarrow \mathbb{R}$  and  $t_i, u_i \in [a, b]$  with  $\delta_n(t) \equiv \delta(t)$ , the Henstock integral for the function  $f$  is obtained.

The Generalized Sequential Henstock integral generalizes some well known Henstock and Sequential Henstock integrals which have been extensively studied by Paxton (2016), Iluebe and Mogbademu (2022). Also, this integral can possibly play an important role in the solution of a generalized linear ordinary differential equation.

**Definition 1.9** Let  $F : [a,b] \times [a,b] \rightarrow \mathbb{R}$  and let  $E \subseteq [a,b]$ :

(a) The function  $F$  is absolutely continuous in the restricted sense on  $E$  ( $F$  is  $AC^*$  on  $E$ ) if  $F$  is bounded on an interval that contains  $E$  and for each  $\varepsilon > 0$ , there exists a sequence of positive gauge function  $\{\delta_n(t)\}_{n=1}^\infty$  on  $[a,b]$  such that

$$\sum_{n=1}^{m_n \in \mathbb{N}} \omega(F, [c_{in}, d_{in}]) < \varepsilon$$

where  $\omega(F, [c_i, d_i]) = \sup\{|F(y) - F(x)| : c_i \leq x \leq y \leq d_i\}$  where  $\{[c_{in}, d_{in}] : 1 \leq i \leq n\}$  is a finite collection of non-overlapping intervals that have endpoints in  $E$  and satisfy  $\sum_{i=1}^{m_n \in \mathbb{N}} (d_{in} - c_{in}) < \delta_n$  for all  $n$ .

(b) The function  $F$  generalized absolutely continuous in the restricted sense on  $E$  ( $F$  is  $AC^*$  on  $E$ ) if  $F|_E$  is continuous on  $E$  and  $E$  can be written as the union of countable sets on each of which  $F$  is  $AC^*$ .

**Remark 1.10** Suppose  $F(t_{in}, u_{in}) = f(t_{in})u_{in}$  where  $f : [a,b] \rightarrow \mathbb{R}$  and  $t_{in}, u_{in} \in [a,b]$  with  $\delta_n(t) \equiv \delta(t)$  in Definition 1.9, we obtain Definition 2.3 of Gordon (1994).

**Theorem 1.11**  $F : [a,b] \times [a,b] \rightarrow \mathbb{R}$  is Generalized Sequential Henstock integrable on  $[a,b]$  if and only if there exist an  $ACG^*$  function  $K$  on  $[a,b]$

$$(GSH) \int_a^x F = K(x) - K(a) \text{ such that } K = F \text{ almost everywhere and any } x \in [a,b].$$

**Remark 1.12** Theorem 1.11 is an analog of Theorem 2.4 of Ding and Ye[2] when  $F(t_{in}, u_{in}) = f(t_{in})u_{in}$ , where  $f : [a,b] \rightarrow \mathbb{R}$  and  $t_{in}, u_{in} \in [a,b]$  with  $\delta_n(t) \equiv \delta(t) \forall n \in \mathbb{N}$ .

The following Lemma 1.13 is useful in our main result.

**Lemma 1.13** If  $F$  is Generalized Sequential Henstock integrable on  $[a,b]$  with the primitives  $K$  and  $v$  is of bounded variation on  $[a,b]$  then the product  $Fg$  is Generalized Sequential Henstock integrable and

$$(GSH) \int_a^b F(t)v(t) = K(b)v(b) - K(a)v(a) - (GSH) \int_a^b K(t)dv(t) \quad (8)$$

where the integral on the right is the Stieltjes case.

The proof of Lemma 1.13 is an analog of the Theorems of Ganguli and Mukherjee (2001) when

$F(t_{in}, u_{in}) = f(t_{in})u_{in}$  where  $f: [a,b] \rightarrow \mathbb{R}$  and  $t_{in}, u_{in} \in [a,b]$  with  $\delta_n(t) \equiv \delta(t)$ , so we omit the proof.

## 2 Main Results

We first state and prove the following lemma which is useful in the proof of our main theorems.

**Lemma 2.1** Let  $F: [a,b] \times [a,b] \rightarrow \mathbb{R}$  be Generalized Sequential Henstock integrable ( $GSH[a,b]$ ),  $v: [a,b] \rightarrow \mathbb{R}, \xi > 0$  be a continuous function,  $r(t)$  be of bounded variation and nonnegative on  $a \leq t \leq s \leq b$ . If there is a constant  $\xi$  such that

$$F(t) \leq \xi + \int_0^t r(s)F(s)dv(s) \quad \text{for } t \in [a,b] \quad (9)$$

holds, where  $\xi > 0$  is a constant. Then

$$F(t) \leq \xi \exp\left(\int_a^t r(s)dv(s)\right) \quad \text{for } t \in [a,b]. \quad (10)$$

Sequel to this, if (9) holds, then (10) does holds.

### Proof

Define

$$r(t) = \int_a^t r(s)F(s)dv(s) \quad (11)$$

Then equation (9) becomes

$$F(t) \leq \xi + r(t) \tag{12}$$

By Theorem 1.11, we have

$$r'(t) = F(s)r(s)_{s \in [a,b]}. \tag{13}$$

Multiply both sides of equation (12) by  $r(t) \exp(-\int_a^t r(s)dv(s))$ , we get

$$F(t)r(t)\exp(-\int_a^t r(s)dv(s)) \leq (\xi + v(t))r(t)\exp(-\int_a^t r(s)dv(s)). \tag{14}$$

That is

$$\begin{aligned} F(t)r(t)\exp(-\int_a^t r(s)dv(s)) - v(t)r(t)\exp(-\int_a^t r(s)dv(s)) \\ \leq \xi r(t)\exp(-\int_a^t r(s)dv(s)) \end{aligned} \tag{15}$$

Combining (15) and (13), we obtain

$$\exp(-\int_a^t r(s)dv(s))(v'(t) - r(t)v(t)) \leq \xi r(t)\exp(-\int_a^t r(s)dv(s)) \tag{16}$$

where  $\exp(-\int_a^t r(s)dv(s))$  is pointwise differentiable and  $ACG^*$ .

By Lemma 1.13,  $\exp(-\int_a^t r(s)dv(s))r(t)$  and  $\exp(-\int_a^t r(s)dv(s))r(t)v(t)$  are

Generalized Sequential Henstock integrable, where  $v(t)$  is  $ACG^*$  and  $r(t)$  is of bounded variation. So both sides of (16) are Generalized Sequential Henstock integrable.

By integrating (16) from  $a$  to  $t$ , we obtain

$$\int_a^t (\exp(-\int_0^t r(s)dv(s)))(v)' dt \leq \xi(1 - \exp(-\int_0^t r(s)dv(s)))$$

Consequently,

$$\exp(-\int_a^t r(s)dv(s))r(t) - r(0) \leq \xi(1 - \exp(-\int_a^t r(s)dv(s)))$$

Which implies

$$v(t) - v(0) \leq \xi(1 - \exp(-\int_a^t r(s)dv(s)))\exp(\int_a^t r(s)dv(s))$$

Simplifying further, we obtain

$$v(t) - v(0) \leq \xi \exp(\int_a^t r(s)dv(s) - 1) \tag{17}$$

. The inequality (17) gives

$$\xi + v(t) \leq \xi \exp(\int_a^t r(s)dv(s)) \tag{18}$$

holds for every  $t \in [a,b]$  and  $v(0) = 0$ . Now combining (18), (11) and (9), we obtain (10). But if (9) hold, then in the same way, we obtain (10)

**Remark 2.2** The Gronwall Bellman’s inequality in Lemma 1.3 follows from Lemma 2.1. Hence, it holds under the Generalized Sequential Henstock integral.

**Theorem 2.3** Let  $r(t),v(t)$  be a bounded variation on  $[a,b]$ ,  $\xi \geq 0$ ,  $r(t) \geq 0$  for every  $s,t \in [a,b]$  and let  $F : [a,b] \times [a,b] \rightarrow \mathbb{R}$  be Generalized Sequential Henstock integrable on  $[a,b]$ , such that for every  $t \in [a,b]$

$$F(t) \leq \xi + \int_a^t r(s)F(s)dv(s) \text{ for } s,t \in [a,b] \tag{19}$$

then

$$F(t) \leq \xi \exp(-\int_a^t r(s)dv(s)) \text{ for } s,t \in [a,b] \tag{20}$$

**Proof.**

Define

$$x(t) = \int_a^t r(s)F(s)dv(s) \tag{21}$$

where  $F(s)$  is a Generalized Sequential Henstock integrable functions  $r(t)$  and  $v(s)$  are functions of bounded variation, so the Generalized Sequential Henstock integral on the right side of (20) exists. By Theorem 1.5, we have

$$x'(t) = r(t)v'(t)F(t) \tag{22}$$

Combining (18) and (20), we have

$$F(t) \leq \xi + x(t) \tag{23}$$

Thus,

$$x'(t) \leq r(t)(\xi + x(t))v'(t) \leq r(t)x(t)v'(t) + \xi v'(t)r(t) \tag{24}$$

since  $r(t)v'(t) \geq 0$ .

That is

$$x'(t) - r(t)v'(t)x(t) \leq \xi v'(t)r(t) \tag{25}$$

Multiplying both sides of (22) by  $\exp(-\int_a^t r(s)dv(s))$ , we obtain

$$\begin{aligned} [\exp(-\int_a^t r(s)dv(s))x(t)]' &\leq \xi v'(t)r(t)\exp(-\int_a^t r(s)dv(s)) \\ &= -\xi[\exp(-\int_a^t r(s)dv(s))] \end{aligned} \tag{26}$$

integrating both sides of (23) from  $a$  to  $t$ , we obtain

$$[\exp(-\int_a^t r(s)dv(s))x(t)] - x(0) \leq \xi[1 - \exp(-\int_a^t r(s)dv(s))]$$

Which implies

$$v(t) \leq \xi(1 - \exp(-\int_a^t r(s)dv(s)))\exp(\int_a^t r(s)dv(s))$$

Simplifying, we obtain

$$x(t) \leq \xi[\exp(\int_a^t r(s)dv(s)) - 1]$$

where  $x(0) = 0$ . Merging (25) and (22), we obtain (20).

**Corollary 2.4** Suppose that  $r(t)$  is a nonnegative Gronwall Bellman's inequality type function on  $[a,b]$  and assume that  $\xi_1, \xi_2 > 0$ . Then, for every  $t \in [a,b]$ ,

$$r(t) \leq \xi_1 + \xi_2 \int_a^t r(s)dv(s) \text{ for } t \in [a,b].$$

implies

$$r(t) \leq \xi_1 \exp(\xi_2(r(t) - r(a))).$$

**Proof.** The proof follows easily from Theorem 2.3.

Next, we will give our main results

**Theorem 2.5** Let  $r(t)$  and  $v(t)$  be a bounded variation on  $[a, b]$ ,  $r(t), v(t) \geq 0$  for every  $t \in [a, b]$ , and let  $F : [a, b] \times [a, b] \rightarrow \mathbb{R}$  be Generalized Sequential Henstock Stieltjes integrable with respect to  $v(t)$  on  $[a, b]$ , if there is a bounded variation function  $h(t)$  such that

$$F(t) \leq h(t) + \int_a^t r(s)F(s)dv(s) \text{ for } s, t \in [a, b]. \tag{27}$$

then

$$F(t) \leq h(t) + \int_a^t r(s)h(s)\exp(-\int_a^s r(s)dv(s))ds, \tag{28}$$

for  $t \in [a, b]$ . Hence, if (25) holds, then as does (27).

**Proof** Define

$$x(t) = \int_a^t r(s)F(s)dv(s)$$

then (26) be

$$F(x) \leq h(t) + x(t). \tag{29}$$

Applying Theorem 1.11 and combining (28) and (29), we obtain

$$x'(t) = r(t)v'(t)F(t) \leq r(t)v'(t)(h(t) + x(t)). \tag{30}$$

That is

$$x'(t) - r(t)v'(t)x(t) \leq r(t)v'(t)h(t). \tag{31}$$

Hence, by multiplying  $\exp(-\int_a^t r(s)dv(s))$  to both sides of (31), we obtain

$$[x(t)\exp(-\int_a^t r(s)dv(s))] \leq r(t)h(t)v'(t)\exp(-\int_a^t r(s)dv(s)).$$

Then, integrating both sides of (31) from  $a$  to  $t$ , we have

$$x(t)\exp(-\int_a^t r(s)dv(s)) \leq \int_a^t r(s)h(s)\exp(-\int_0^t r(s)dv(s))$$

Hence,

$$x(t) = \int_a^t r(s)h(s)\exp(\int_a^t r(s)dv(s)) \tag{32}$$

By adding  $h(t)$  to both sides of (32), we obtain (28).

**Remark 2.6** In Ding and Ye (2009), the author obtained similar type of inequalities in Theorems 3.3 and 3.4. In this paper, we relax the conditions of the existence of these inequalities and use the Generalized Sequential Henstock integral theory to prove these inequalities. Hence, we can use our results to study the stochastic differential equation.

The following result follows for the Stieltjes type Sequential Henstock integrable function in which the integrator is dominated by two continuous functions, defined below

Consider the integral of a continuous function  $f: [a,b] \rightarrow \mathbb{R}$

$$\int_a^b f dg \tag{33}$$

Suppose  $g$  is a non-decreasing product function define by

$$g = r(x).s(x)$$

then

$$dg = rds + sdr$$

therefore equation (1) becomes

$$\int_a^b f d(rs) = \int_a^b f(rds + sdr) = \int_a^b f(rds) + \int_a^b f(sdr)$$

Now, setting  $\Delta = rs$ ,  $\Delta_s = rds$  and  $\Delta_r = sdr$  then

$$\int_a^b f d(\Delta) = \int_a^b f(\Delta_s) + \int_a^b f(\Delta_r)$$

Suppose  $f$  is a Sequential Henstock integrable function, we give the following new definition

**Definition 2.7** Let  $g : [a,b] \rightarrow \mathbb{R}$  be a non decreasing function. A function  $f : [a,b] \rightarrow \mathbb{R}$  is Stieltjes dominated type Sequential Henstock integrable with respect to  $g$  on  $[a,b]$  to a number  $\int_a^b f d\Delta$  if for any  $\varepsilon > 0$  there exists a sequence of positive gauge functions  $\{\delta_n(x)\}_{n=1}^\infty$  such that

$$\left| \sum_{i=1}^{m_n} f(t_{i_n}) [\Delta(u_{i_n}) - \Delta(u_{(i-1)_n})] - \int f d\Delta \right| < \varepsilon,$$

whenever  $P_n = \{(u_{(i-1)_n}, u_{i_n}), t_{i_n}\}_{i=1}^{m_n}$  is a  $\delta_n(x)$ -fine partitions of  $[a,b]$ .

We use  $STHS_f[a,b]$  to denote the set of all Stieltjes dominated type Sequential Henstock integrable functions defined on  $[a,b]$ .

### 3 Application

We shall give some application of Theorem 2.2 and 2.4 to stochastic differential equation. Some properties of solution of initial value problems will be considered.

Let  $G \times G \subset \mathbb{R}^{N+1}$  be an open set. Suppose  $F : G \times G \rightarrow \mathbb{R}^{N+1}$  is a given

$\mathbb{R}^{N+1}$  valued function  $F(u_{in}, t_{in})$  defined for  $(u_{in}, t_{in}) \in G, u_{in} \in \mathbb{R}^n, t_{in} \in \mathbb{R}$ .

**Definition 3.1** A function  $x : [a,b] \times [a,b] \rightarrow \mathbb{R}^N$  is called the solution of the stochastic differential equation (SDE)

$$\frac{dx(t)}{dt} = a(t)x(t), x(0) = x_0 \tag{34}$$

on  $[a,b]$  if  $a(t)$  is a non-deterministic or stochastic parameter given as

$$a(t) = F(t) + h(t)\xi(t), \tag{35}$$

for all  $t \in [a,b]$  and if

$$x_{s_2}(t) = x_{s_1}(t) + \int_0^t F(s, x(s)) ds, \tag{36}$$

holds for every  $s_1, s_2 \in [a,b]$  the identity.  $\xi(t)$  denotes a white noise process and the integral on the right hand side of (35) should be understood as Generalized Sequential Henstock integral on  $[a,b] \subset \mathbb{R}$

When considering the solution of the stochastic differential equation, we have the following lemma.

**Lemma 3.2** Let  $F : [a,b] \times [a,b] \rightarrow \mathbb{R}$  If a function be a Generalized Sequential Henstock integrable function. Suppose  $x : [a,b] \times [a,b] \rightarrow \mathbb{R}^n$  is a solution of the stochastic differential equation on  $[a,b]$  then for every  $\lambda \in [a,b]$ , we have

$$x(s) = x(\lambda) + \int_s^\lambda F(s, x(s)) ds, \quad s \in [a, b] \tag{37}$$

If a function  $x : [a,b] \times [a,b] \rightarrow \mathbb{R}^n$  satisfies the integral equation (35) for some  $\lambda \in [a,b]$  and all  $s \in [a,b]$ , then  $x$  is a solution of the stochastic differential equation (32).

Now, we define the following type of stochastic differential equation by

$$H(x, t) = \int_t^{t_0} F(s, x(s)) dv(s), \quad s \in [a, b] \tag{38}$$

where  $v : [a,b] \times [a,b] \rightarrow \mathbb{R}$  is bounded variation on compact subintervals of  $\mathbb{R}$ ,  $F : [a,b] \times [a,b] \rightarrow \mathbb{R}$  is Generalized Sequential Henstock Stieltjes integrable with respect to  $v(t)$ . Considering the solution of the stochastic differential equation

$$\frac{dx(t)}{dt} = DH(x, t) \tag{39}$$

Hence, from Lemma 3.2,  $x(t)$  is a solution of the integral of (37) through  $(x_0, t_0)$  on  $[a,b]$  where  $t_0 \in [a,b]$  and  $x(t_0) = x_0$ , if and only if it satisfies the Generalized Sequential Henstock integrable equation

$$x(t) = x_0 + \int_t^{t_0} F(s, x(s)) dv(s), \quad t \in [a, b] \tag{40}$$

Hence, we can use our results to prove the uniqueness of the solution of (37) under the assumption that  $F$  satisfies the Lipschitz condition, i.e

$$|F(x(t), t) - F(y(t), t)| \leq \alpha(t) |x(t) - y(t)| \tag{41}$$

for every  $t \in [a,b]$  and  $\alpha(t) \geq 0$ , where  $\alpha(t)$  is Generalized Sequential Henstock integrable on  $[a,b]$ .

Let  $x(t)$  and  $y(t)$  be two solutions of (37) through the same point  $(x_0, t_0)$ , i.e.,  $x(t_0) = x_0 = y(t_0)$ . Let  $z(t) = |x(t) - y(t)|$ , clearly  $z(t_0) = 0$  and from equation (37), we have

$$\begin{aligned}
 z(t) &= |(GSH) \int_t^{t_0} (F(s, x(s)) - F(s, y(s)))dv(s)| \\
 &\leq (GSH) \int_t^{t_0} \alpha(s)|x(s) - y(s)|dVar_{t_0}^s v,
 \end{aligned}
 \tag{42}$$

where  $Var_{t_0}^s v$  is the total variation function of  $v$  on  $[t_0, s]$ .  $x(t), y(t)$  satisfy (37) and  $F(t, x(t))$  and  $F(t, y(t))$  are Generalized Sequential Henstock integrable on  $[t_0, t]$  with respect to  $v(t)$ . where  $v(t)$  is an identity function of bounded variation.  $x_0$  is a constant. So  $x(t)$  and  $y(t)$  are continuous  $\alpha(t) \geq 0$  is Generalized Sequential Henstock integrable. Clearly  $(GSH) \int_t^{t_0} \alpha(s)|x(s) - y(s)|dVar_{t_0}^s v$  exists by the hypothesis of  $\alpha(t)$ , The function  $Var_{t_0}^s v$  is non-decreasing in  $s$  on  $[t_0, t]$  so  $Var_{t_0}^s \alpha(t) \geq 0$ . Using Theorem 2.3, with  $\xi \equiv 0$  and  $k(t) = \alpha(t)$ , we get

$$z(t) \geq 0 \cdot \exp \int_a^t \alpha(s) dVar_{t_0}^s v.$$

That is

$$|x(t) - y(t)| \geq 0. \tag{43}$$

Thus,  $x(t) = y(t)$ . Thus, we have proved the uniqueness of the solution.

**References**

Bellman, R. (1953) Stability Theory of Differential equations. McGraw-Hill, New York, , 79-85.

Ding, X. F. and Ye, G. J. (2009), Generalized Gronwall-Bellman Inequalities Using the Henstock-Kurzweil Integral. *Southeast Asian Bulletin of Mathematics*. 33, 703-713.

Ganguly, D. K., and Mukherjee R. (2001), The Generalised Approximate Perron Integral. *Mathematica Slovaca*, 58(1), 31-42.

Gong, Z. A (2003) New Discriptive Characterization of Henstock-Kurzweil Integrals. *Southeast Asian Bull.* 27(3),445-450.

Gordon, R. (1994), The Integral of Lebesgues,

Denjoy, Perron and Henstock. *Graduate studies in Mathematics. American Mathematical Society, Providence, RI*, 4. 1-5.

Gronwall, T. H. (1919), Notes on the derivatives with respect to a parameter of the solutions of a systems of Differential Equations. *Ann. of Math.* 20, 292-296.

Hamid, M. E; Xu, L. and Gong, Z. The (2017), Henstock-Stieltjes Integral For Set Valued Functions. *International Journal of Pure and Applied Mathematics.* 114(2), 261-275.

Herzog, F. (2013), Stochastic Differential Equations. *Stochastic Systems*, 1-64.

Iluebe V. O, and Mogbademu A. (2022), A. On ap-Sequential Henstock Integral for Interval valued Functions. *Int. J. Nonlinear Anal. Appl.* 13(2), 3095-3103.

Iluebe V. O, and Mogbademu A. A. (2023),

General Sequential Topological HenstockType Integral. *Acta Universitatis Apulences.* 74, 85-96.

Iluebe V. O. and Mogbademu A. A. (2022), On Sequential Henstock Stieltjes Integral For  $L^p[0,1]$ -Interval Valued Functions. *Bull. Int. Math. Virtual Inst.*, 12, 369 - 378.

Iluebe, V. O. and Mogbademu A. A. (2022), Convergence Results for Sequential Henstock Stieltjes Integral in Real Valued Space. *TJMM*, 14(2), 169 - 176.

Iluebe, V. O. and Mogbademu, A. A. (2023), Equivalence Between Sequential Henstock And Topological Henstock Integral For Interval valued Functions. *J. Mahani Math. Res.* 12(2), 267 - 274.

Mogbademu A. A., Iluebe V. O. and Bishop, S. A. (2025), On General Sequential Henstock-Type Integral For Locally Convex Spaced-valued Functions. *Journal of the Indian Math. Soc.* 92.(1), 38-49.