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Root systems and Weyl group actions on flag varieties

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Abstract. This article gives a review of some results on the interplay between root systems, Weyl groups, and the geometry of flag varieties, highlighting their connections through Schubert calculus and representation theory. The results illustrate how the cohomological and combinatorial structures arising from these objects provide a unified framework for geometric representation theory.

Keywords: Root systems, Weyl groups, Flag varieties, Schubert varieties, Cohomology, Geometric representation theory, Algebraic groups, Bruhat order.
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1. Introduction and Background

The study of symmetry in algebra and geometry finds one of its most complete realizations in the theory of *root systems* and their associated *Weyl groups*. Originating from the work of Cartan and Weyl [1939, 1909] in the classification of semi simple Lie algebras, root systems encode reflection symmetries in Euclidean space and provide a bridge between abstract algebraic structures and concrete geometric configurations. Each semi simple Lie algebra $\mathbf{g}$ possesses a decomposition relative to a maximal torus T, giving rise to a finite set of roots in the dual space t^*. The reflections defined by these roots generate the *Weyl group* W, which acts discretely on t^* and preserves the root system. This combinatorial and geometric structure determines the representation theory of $\mathbf{g}$ and the geometry of its associated algebraic group G. On the geometric side, the *flag variety* G/B—where B is a Borel subgroup—serves as a projective homogeneous space on which W acts by symmetries. Its decomposition into Schubert cells encodes the combinatorics of W through the *Bruhat order*, establishing a direct correspondence between algebraic data and geometric stratification. Historically, the connection between these two realms was clarified by Borel [1953], Chevalley [1955], and Demazure [1973]. Modern developments extend these ideas to the study of quantum cohomology, Kac–Moody groups, and geometric representation theory (see Brion [2005], and Kumar [2002]). The goal of this paper is to present a detailed synthesis of the algebraic and geometric aspects of root systems, Weyl group actions, and flag varieties. We shall establish the basic properties, provide explicit examples, and highlight their cohomological and representation-theoretic significance. The organization of this paper is as follows: Section 2 introduces root systems and proves their key properties. Section 3 develops the theory of Coxeter and Weyl groups with examples. Section 4 examines flag varieties and their decomposition. Section 5 explores cohomology and Schubert calculus, while Section 6 connects these ideas to representation theory. Sections 7–8 present explicit computations and concluding remarks.

2. Root Systems: Definition, Examples, and Basic Properties

In this section, we give definitions of some well-known basic terms with examples which shall be needed in the sequel.

We recall standard definitions and results on root systems; details may be found in Humphreys [1990].

Definition 2.1. Let V be a finite-dimensional real Euclidean space with inner product $\langle\cdot,\cdot\rangle$. A *(reduced, crystallographic) root system* is a finite subset $\Phi \subset V \setminus \{0\}$ satisfying:

(i) Φ spans V

(ii) If $\alpha \in \Phi$, then the only scalar multiples of α in Φ are $\pm\alpha$.
(iii) For each $\alpha \in \Phi$, the reflection

$$s_\alpha(v) = v - 2\frac{\langle v,\alpha\rangle}{\langle \alpha,\alpha\rangle}\alpha$$

preserves Φ
(iv) For all $\alpha, \beta \epsilon \Phi$, the Cartan integer

$$\langle \beta, \alpha^\vee\rangle = 2\frac{\langle \beta,\alpha\rangle}{\langle \alpha,\alpha\rangle}$$

is an integer.

The integer $\langle \beta, \alpha^\vee\rangle$ is called the Cartan integer. These conditions ensure that Φ is symmetric and highly constrained, leading to a small number of irreducible types.

Example 2.2 (Type A_2 [9, §2.1]). Let

$$H = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 0$$

with the inner product induced from $\mathbb{R}^3$. Define

$$\Phi = \{\pm(e_i - e_j) \mid 1 \leq i < j \leq 3\} \subset \boldsymbol{H}$$

Then Φ is a root system of type A_2 consisting of six roots arranged at angles of 60°.

Example 2.3 (Type B_2). In $V = \mathbb{R}^2$, define

$$\Phi = \{\pm e_1, \pm e_2, \pm(e_1 + e_2), \pm(e_1 - e_2)\}.$$

Then Φ forms a root system of type B_2 consisting of four long and four short roots, symmetric under reflections across the coordinate axes and diagonals.

Lemma 2.4 (Reflection Invariance). Humphreys [1990] Let Φ be a root system in V. Then for each $\alpha \in \Phi$, the reflection s_α defined above is an orthogonal transformation, and $s_\alpha(\Phi) = \Phi$.

Proposition 2.5 (Finiteness and Symmetry). Humphreys[1990] If Φ is a root system in V, then:
(i) Φ is finite and spans V.
(ii) For each $\alpha \in \Phi$, both α and $-\alpha$ belong to Φ.
(iii) The group generated by all reflections $s_\alpha(\alpha \in \Phi)$ is finite.

Remark 2.6. Each root system decomposes into *positive* and *negative* roots once a choice of hyperplane is fixed. The selection of a basis of simple roots $\Delta = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ determines all structural constants and defines the associated *Dynkin diagram.*

3. Coxeter and Weyl Groups: Structure, Relations, and Examples

Given a root system Φ in a Euclidean space V , we associate to each root $\alpha \in \Phi$ the reflection

$$s_\alpha(v) = v - 2\frac{\langle v,\alpha\rangle}{\langle \alpha,\alpha\rangle}\alpha$$

The group generated by these reflections plays a central role in both the algebraic and geometric theory of Lie groups.

Definition 3.1 (Weyl Group). The Weyl group $W(\Phi)$ associated to a root system Φ is the subgroup of $O(V)$ generated by all reflections s_α for $\alpha \in \Phi$, i.e

$$W(\Phi) = \langle s_\alpha \mid \alpha \in \Phi \rangle.$$

Each s_α is orthogonal, and $W(\Phi)$ acts on V by permuting the roots. The resulting action is faithful, and the group $W(\Phi)$ is finite because Φ is finite.

Definition 3.2 (Coxeter System). A *Coxeter system* is a pair (W, S) where W is a group generated by a finite set $S = \{s_1, s_2, \ldots, s_n\}$ subject to relations

$$(s_i s_j)^{m_{ij}} = 1$$

where $m_{ii} = 1$ and $m_{ij} = m_{ji} \in \{2, 3, 4, 6, \infty\}\, for\ i \neq j$. The integers m_{ij} are called the *Coxeter exponents.*

Remark 3.3. Every Weyl group $(W(\Phi), S)$ of a root system is a Coxeter system, where $S = \{s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_n}\}$ corresponds to reflections in the simple roots. The relations among the generators encode the geometry of the root system and can be represented by a *Coxeter diagram*, which is equivalent to the *Dynkin diagram* in the crystallographic case.

Example 3.4 (Weyl Group of Type A_2). Humphreys[1990] Let Φ be the root system of type A_2. The simple roots are $\alpha_1 = e_1 - e_2$ and $\alpha_2 = e_2 - e_3$. The corresponding reflections satisfy

$$(s_{\alpha_1} s_{\alpha_2})^3 = 1, \qquad {s_{\alpha_i}}^2 = 1$$

Hence the Weyl group has presentation

$$W(A_2) = \langle s_1,\ s_2 \mid {s_1}^2 = {s_2}^2 = (s_1\, s_2)^3 = 1\rangle$$

which is isomorphic to the symmetric group S_3. Geometrically, $W(A_2)$ acts on the plane of roots by permuting the vertices of a regular hexagon.

Example 3.5 (Weyl Group of Type B_2). [13] For $\Phi = B_2$, take simple roots $\alpha_1 = e_1 - e_2$ (long) and $\alpha_2 = e_2$ (short). Then the reflections satisfy

$$(s_{\alpha_1} s_{\alpha_2})^4 = 1.$$

Thus

$$W(B_2) = \langle s_1,\ s_2 \mid {s_1}^2 = {s_2}^2 = (s_1\, s_2)^4 = 1\rangle$$

This is the dihedral group of order 8, corresponding to the symmetries of a square. The distinction between long and short roots manifests in the unequal lengths of α_1 and α_2.

Proposition 3.6 (Coxeter Relations for Weyl Groups). Let Φ *be a reduced, crystallographic root system with set of simple roots* $\Delta = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. *Then the Weyl group* $W(\Phi)$ *satisfies the Coxeter relations:*

$$(s_i s_j)^{m_{ij}} = 1$$

Where $m_{ij} = \pi / \theta_{ij}$ *and* θ_{ij} *is the angle between* α_i *and* α_j. *The Coxeter matrix* (m_{ij}) *determines the entire structure of* $W(\Phi)$.

Corollary 3.7. *Every Weyl group is a finite reflection group acting discretely on* V*. Its fundamental domain is a simplicial cone called the* Weyl chamber.

Remark 3.8. The Weyl chambers partition V into regions corresponding to the orbits of W. Each chamber corresponds to a distinct choice of positive roots. This geometric structure underlies the Bruhat decomposition of the flag variety G/B.

4. Flag Varieties and Weyl Group Actions

In this section, we develop the geometry of flag varieties and then show how Weyl group combinatorics appears naturally through the Bruhat decomposition and the Schubert cell stratification. We include proofs (or sketches, where the full details are standard) and an explicit example of type A.

4.1. **Flag varieties as homogeneous spaces**. Let G be a connected complex semi-simple algebraic group, $B \subset G$ a Borel subgroup, and $T \subset B$ a fixed maximal torus. The *flag variety* of G is the projective homogeneous variety

$$\mathcal{F} := G/B.$$

Points of $\mathcal{F}$ are in bijection with Borel subgroups of G; explicitly $gB \in G/B$ corresponds to the conjugate Borel subgroup gBg^{-1}.

Remark 4.1. When $G = GL_n(\mathbb{C})$ $(or\ SL_n(\mathbb{C}))$, the flag variety G/B is the variety of complete flags

$$\{0\} = V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = \mathbb{C}^n, \quad such\ that, dim\ V_i = i.$$

4.2. **Bruhat decomposition.** Let $N_G(T)$ be the normalizer of T in G and $W := N_G(T)/T$ the Weyl group. Fix lifts $\dot{w} \in N_G(T)$ for each w ∈W. The double coset decomposition of G with respect to B is the *Bruhat decomposition:*

Theorem 4.2 (Bruhat decomposition). *[2]*

$$\mathrm{G} = \bigcup_{w\in W} B\dot{w}\,\mathrm{B}.$$

Passing to the quotient G/B *gives a decomposition*

$$\mathcal{F} = G/B = \bigcup_{w\in W} C_w, \qquad C_w := B\dot{w}B/B.$$

Each C_w *is a locally closed, B-stable subset of* $\mathcal{F}$. Adetunji *[2024]*

4.3. **Schubert cells and varieties.**

Definition 4.3. Adetunji [2024] For $w \in W$, the Schubert cell C_w is $\mathrm{B}\dot{w}\mathrm{B}/\mathrm{B} \subset G/B$. Its Zariski closure

$$X_w := \overline{C_w}$$

is the Schubert variety associated to *w*.

Proposition 4.4. Brion [2005] *Each Schubert cell* C_w *is isomorphic (as a complex variety) to an affine space* $\mathbb{C}^{\ell(w)}$*where* $\ell(w)$ *is the length of w in the Coxeter generators. Consequently* $dim_{\mathbb{C}} X_w = \ell(w)$.

Corollary 4.5. Brion [2005] The Schubert varieties X_w provide a cell decomposition of $\mathcal{F}$; in particular the classes $[X_w]$ form an additive $\mathbb{Z}$-basis of $H^*(\mathcal{F}, \mathbb{Z})$.

4.4. **Bruhat order and inclusions.** The closure relations among Schubert varieties are encoded by the Bruhat order on W:

Proposition 4.6. For $v, w \in W$,

$$X_v \subseteq X_w \Leftrightarrow v \leq w$$

in the Bruhat (strong) order.

4.5. **Fixed points and tangent weights.** The torus T acts on $\mathcal{F}$ with finitely many fixed points, namely the cosets $\{ \dot{w}B \mid w \in W\}$. The tangent space at a fixed point is described in terms of root spaces:

Proposition 4.7. At the T-fixed point $\dot{w}B \in \mathcal{F}$,

$$\mathrm{T}_{\dot{w}\mathrm{B}}\mathcal{F} \cong \bigoplus_{\alpha\in\phi^+} \mathfrak{g}_{-\mathfrak{w}(\alpha)}$$

where $\mathfrak{g}\beta$ denotes the root space for β.

The description in 4.7, shows that the weights (roots) that occur in the tangent space at $\dot{w}B$ are exactly the roots sent by w from positive to negative, and hence $dimC_w$equals the number of such roots, i.e., $\ell(w)$.

Example 4.8. Full flag variety for $SL_3(\mathbb{C})$.
Let $G = SL_3(\mathbb{C})$, B the upper-triangular Borel, and T the diagonal torus. The Weyl group $W \cong S_3$ has six elements with lengths 0,1,1,2,2,3. The flag variety G/B parameterizes complete flags $0 \subset V_1 \subset V_2 \subset \mathbb{C}_3$.
Schubert cells correspond to permutations; for instance:

$$C_{s_1} = (V_1, V_2) : V_1 \subset \langle e_2, e_3 \rangle, \quad V_2 = \langle e_1, e_2 \rangle \qquad (suitably) \cong \mathbb{C},$$

and similarly for other cells. Closure relations reproduce the Bruhat Hasse diagram of S_3. This concrete model is useful for explicit calculations in Schubert calculus (see Fulton [1997]).

5. Cohomology and Schubert Calculus

The geometry of the flag variety $\mathcal{F} = \mathrm{G/B}$ is intimately tied to the combinatorics of the Weyl group W. The cohomology ring $H^*(\mathcal{F}, \mathbb{Z})$ carries a distinguished basis given by the fundamental classes of Schubert varieties, and the multiplication structure constants encode deep representation-theoretic data.

5.1. **Borel's description of Cohomology**.

Theorem 5.1 (Borel, [1953]). *Let G be a connected, complex semi-simple algebraic group with maximal torus T and Weyl group W. Then there is an isomorphism of graded rings:*

$$H^*(G/B, \mathbb{Q}) \cong S(\mathfrak{t}^*)/\langle S(\mathfrak{t}^*)_+^W \rangle$$

where $S(\mathfrak{t}^*)$ is the symmetric algebra on $(\mathfrak{t}^*)$ and $S(\mathfrak{t}^*)_+^W$ denotes the ideal generated by homogeneous W- invariant polynomials of positive degree.

Remark 5.2. This identification makes $H^*(G/B)$ isomorphic to the *coinvariant algebra* of W, a central object in geometric representation theory. The natural action of W on $\mathfrak{t}^*$ induces the regular representationof W on this quotient.

5.2. **Schubert basis.** Let $\{[X_w]\}_{w\in W}$ denote the fundamental cohomology classes of Schubert varieties.
Proposition 5.3. Brion [2005] *The set* $\{[X_w] \mid w \in W\}$ *forms a graded additive basis of* $H^*(G/B,\mathbb{Z})$*, satisfying*

$$deg[X_w] = 2\ell(w).$$

5.3. **Structure constants and the Chevalley formula**. The product structure in $H^*(G/B)$ can be expressed in the Schubert basis:

$$[X_u]\cdot[X_v] = \sum_{w\in W} c_{u,v}^{w}\,[X_w]$$

where the integers $c_{u,v}^{w}$ are the structure constants of the Schubert calculus.

Theorem 5.4 (Chevalley formula, [1955]). Let σ_α denote the Schubert divisor class corresponding to a simple reflection s_α. Then for any $w \in W$,

$$\sigma_\alpha\cdot[X_w] = \sum_{\substack{w\prime\in W\\ w'=ws_\beta,\\ \ell(w')=\ell(w')+1}} \langle\omega_\alpha,\beta^\vee\rangle[X_{w'}],$$

where ω_α is the fundamental weight dual to $\alpha^\vee$.

5.4. **Monk's formula in type** A_{n-1}. In the case $G = GL_n(\mathbb{C})$, the flag variety G/B has Schubert classes σ_w indexed by permutations $w \in S_n$. Monk's formula gives an explicit description of multiplication by a divisor class.

Theorem 5.5 (Monk's formula, [1959]). *For the simple reflection* s_i *corresponding to the transposition* $(i, i\,+\,1)$,

$$\sigma_{s_i}\cdot\sigma_w = \sum_{\substack{w'=w(i,j)\\ i<j,\ell(w')=\ell(w)+1}} \sigma_{w'}\ ,$$

where $w(i,j)$ *denotes the permutation obtained from* w *by swapping* $w(i)$ *and* $w(j)$.

Example 5.6. Cohomology of the full flag variety in type A_2
For $G = SL_3(\mathbb{C})$, the Weyl group $W \cong S_3$ has six elements. The cohomology ring $H*(G/B)$ is generated by divisor classes $\sigma_{s_1}, \sigma_{s_2}$ with relations

$$\sigma_{s1}^2+\sigma_{s_2}^2+\sigma_{s_1\sigma_{s_2}}=0, \qquad \sigma_{s_1}^3=\sigma_{s_2}^3=0.$$

Multiplication is governed by Monk's formula, giving

$$\sigma_{s_1}\cdot\sigma_{s_2} = \sigma_{s_1s_2}+\sigma_{s_2s_1}..$$

Remark 5.7. This reproduces the intersection pairing among Schubert varieties corresponding to the Bruhat order.

5.5. Connections to representation theory.

Theorem 5.8 (Borel–Demazure, [1973]). *The cohomology ring $H^*(G/B)$ is isomorphic, as a graded W-module, to the coinvariant* algebra

$$\mathbb{C}[\mathrm{t}]/\langle \mathbb{C}[\mathrm{t}]_+^W \rangle,$$

and decomposes into irreducible W-representations containing each irreducible representation of W exactly once.

This result provides a deep link between topology, algebra, and representation theory. In particular, the regular representation of the Weyl group arises naturally from the action on the cohomology of the flag variety.

Remark 5.9. The algebraic structure of $H^*(G/B)$ has inspired generalizations such as quantum cohomology and equivariant cohomology, leading to modern results in geometric representation theory (see Brion [2005], Kumar [2002]).

6. Geometric Representation Theory Connections

The deep relationship between the geometry of flag varieties and the representation theory of semi-simple Lie algebras is one of the most powerful ideas in modern mathematics. In this section, we review some key links, including the Borel–Weil theorem, Demazure's geometric realization of highest-weight representations, and the Springer correspondence connecting the Weyl group to the geometry of nilpotent orbits.

6.1. **The Borel–Weil theorem**. Let G be a connected complex semi-simple algebraic group, B a Borel subgroup, and λ a dominant integral weight. Associated to λ is a line bundle $\mathcal{L}_\lambda$ on the flag variety G/B.

Theorem 6.1 (Borel–Weil, [1953]). *For each dominant integral weight λ, there is an isomorphism of G-modules:*

$$H^0(G/B, \mathcal{L}_\lambda) \cong V_\lambda^*,$$

where V_λ is the irreducible representation of G with highest weight λ.

Remark 6.2. This geometric construction realizes every finite-dimensional irreducible representation of G as global sections of a line bundle on the flag variety. It provides a bridge between algebraic geometry and representation theory.

6.2. **Demazure modules and Schubert varieties**. Demazure extended the Borel–Weil theorem by replacing the full flag variety G/B with Schubert varieties X_w.

Definition 6.3 (Demazure module). For a dominant weight λ and $w \in W$, the Demazure module $V_w(\lambda)$ is the B-submodule of V_λ generated by the weight space $V_\lambda^{w\lambda}$.

Theorem 6.4(Demazure, [1973]). *There is an isomorphism of B-modules:*

$$H^0(X_w, \mathcal{L}_\lambda) \cong V_w(\lambda)^*.$$

Remark 6.5. For $w = w_0$ (the longest element in W), $X_{w0} = G/B$, and we recover the Borel–Weil theorem. Thus, Demazure's theorem generalizes Borel–Weil to the partial geometry of Schubert varieties.

6.3. **The Springer correspondence**. One of the most profound connections between geometry and representation theory arises from the Springer resolution. Let $\mathcal{N} \subset \mathfrak{g}$ be the nilpotent cone and define the *Springer resolution*:

$$\widetilde{\mathcal{N}} = \{(x, \mathfrak{b}) \in \mathcal{N} \times \mathcal{F} | x \in \mathfrak{b}\},$$

with projection $\pi: \widetilde{\mathcal{N}} \to \mathcal{N}$.

Theorem 6.6 (Springer, [1976]). For each nilpotent orbit $\mathcal{O} \subset \mathcal{N}$, the top cohomology $H^{2d}(\pi^{-1}(x), \mathbb{C})$ of the corresponding Springer fiber carries an action of the Weyl group W. The resulting representations $H^*\left(\pi^{-1}(x)\right)$ exhaust all irreducible representations of W.

Remark 6.7. This correspondence establishes a deep link between the geometry of the nilpotent cone and the representation theory of W, connecting Lie algebra orbits, geometry of flag varieties, and group theory.

6.4**. Example: Type** A_2. $For\ G = SL\_3(\mathbb{C})$, the nilpotent cone consists of three orbits corresponding to the partitions of 3: $(3), (2,1), and\ (1,1,1)$. The Springer fibers over these orbits have dimensions $0, 1, and\ 2$ respectively. Their top cohomologies realize the trivial, standard, and sign representations of the Weyl group S_3.

6.5. **Modern extensions**. The Springer correspondence has since been generalized]to the geometric *Satake correspondence* and the *Langlands dual group*, providing geometric realizations of representations of reductive groups over local and global fields (see Lusztig [1984], Mirkovic [2007]).

Remark 6.8. These developments show how flag varieties, Schubert varieties, and nilpotent orbits encode deep algebraic and arithmetic structures, unifying geometry and representation theory.

7. Applications and Recent Directions

The rich interplay between root systems, Weyl groups, and flag varieties continues to drive developments in representation theory, algebraic geometry, and mathematical physics. We highlight here several contemporary applications and research directions that extend classical theory into new realms.

7.1. **Geometric Satake correspondence**. One of the central achievements in geometric representation theory is the *Geometric Satake correspondence*, which connects the geometry of the affine Grassmannian to representations of the Langlands dual group.

Theorem 7.1(Geometric Satake, [2007*]). Let G be a connected complex reductive group, and Gr_G its affine Grassmannian. Then the category of* $\mathrm{G}(\mathcal{O})$*-equivariant perverse sheaves on Gr_G (with convolution product) is tensor equivalent to the category of finite-dimensional representations of the Langlands dual group $\widehat{G}$.*

Remark 7.2. This result provides a geometric incarnation of the Tannakian formalism and plays a foundational role in the geometric Langlands program. It extends the link between geometry and representation theory from the finite-dimensional flag variety G/B to infinite-dimensional affine analogues.

7.2. **Quantum groups and canonical bases**. The discovery of quantum groups by Drinfeld and Jimbo provided a deformation of universal enveloping algebras of Lie algebras, tightly related to the geometry of flag varieties.

Definition 7.3. The quantum group $U_q(\mathfrak{g})$ is a Hopf algebra deformation of $U(\mathfrak{g})$ depending on a parameter q. Its representations specialize to classical ones as $q \to 1$.

Theorem 7.4 (Lusztig, [1990]). *The canonical basis (or global crystal basis) for* $U_q(\mathfrak{n}^-)$ *can be constructed geometrically using the intersection cohomology of quiver varieties and the geometry of flag varieties.*

Remark 7.5. This geometric realization provides a bridge between quantum group theory and geometry, enabling deep categorification results in the study of knot invariants and cluster algebras.

7.3. **Equivariant and quantum cohomology**. Equivariant and quantum extensions of Schubert calculus have introduced new algebraic structures and connections to symplectic geometry.

Theorem 7.6 (Kim, [1999]). *The quantum cohomology ring* $QH^*(G/B)$ *is a deformation of* $H^*(G/B)$ *whose structure constants count rational curves in* G/B. *It can be presented as*

$$QH^*(G/B) \cong [\mathrm{t}]^W/\langle f_1(q), \ldots, f_r(q)\rangle,$$

where $f_i(q)$are q-deformations of the fundamental W-invariant polynomials.

Remark 7.7. Quantum cohomology provides a modern setting for enumerative geometry and links representation theory to Gromov–Witten invariants and integrable systems.

7.4. **Categorification and higher representation theory**. Recent work has aimed to categorify the representation-theoretic structures arising from Weyl groups and flag varieties.

Definition 7.8. A *categorification* of a vector space V is an additive or triangulated category $\mathcal{C}$ such that $K_0(\mathcal{C}) \cong V$, where K_0 is the Grothendieck group.

Example 7.9. The category $\mathcal{O}$ for a semi-simple Lie algebra $\mathfrak{g}$ and its highest weight subcategories categorify weight spaces of representations of $\mathfrak{g}$. Similarly, Soergel bimodules categorify the Hecke algebra associated to W.

Theorem 7.10 (Soergel, [1990]). The indecomposable Soergel bimodules $\{B_w\}_{w\in W}$ form a categorification of the Hecke algebra $\mathcal{H}_W$, satisfying:

$$[B_w]\cdot[B'_w] = \sum_{x\in W} h^x_{w,w'}\,[B_x],$$

where $h^x_{w,w'}$ are the *Kazhdan–Lusztig structure constants*.

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.5. **Future directions**. Research at the intersection of these fields continues to evolve rapidly, with several promising directions:

- Connections between *mirror symmetry* and quantum cohomology of flag varieties.

- The study of *Kac–Moody and affine flag varieties* in relation to modular representation theory.
- Extensions of the *geometric Satake equivalence* to derived and motivic settings.
- Applications of *Soergel bimodules* and *categorical actions* in low-dimensional topology.

Remark 7.11. These directions underscore the continuing vitality of the subject: from the classical theory of root systems to modern categorical and geometric frameworks, flag varieties remain central to the landscape of modern mathematics.

8. Conclusion and Acknowledgments

In this paper, we have developed a comprehensive exploration of the interplay between root systems, Weyl groups, and the geometry of flag varieties. Starting from the axiomatic definition of root systems and their reflection symmetries, we examined how Weyl group actions shape the structure of flag varieties and Schubert cells. Through the work of Borel, Chevalley, and Demazure, we review the cohomological and geometric frameworks that connect these objects to the representation theory of semi-simple Lie algebras.

We further discussed how these classical ideas have evolved into the modern language of geometric representation theory. The Springer correspondence, geometric Satake equivalence, and categorification of Hecke algebras by Soergel bimodules illustrate the unifying role played by the geometry of G/B across many branches of mathematics.

Beyond their intrinsic beauty, these structures continue to find new applications — from quantum groups and mirror symmetry to low-dimensional topology and arithmetic geometry. The study of root systems and Weyl group actions thus remains a vibrant field of ongoing research.

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