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Convergence of generalized Mann iteration scheme for some contractive mappings in convex G_{b}- metric spaces with application

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Abstract

In this research, novel results on the generalized Mann iteration scheme to the existence of fixed points of Kannan contraction mappings, Chatterjea contraction mappings and Hardy - Rogers contraction mappings in the framework of complete convex G_{b}-metric spaces, where convexity is defined in the sense of Takahashi, are respectively established and illustrated. These results individually improve upon a similar result that has been established for Banach contraction mappings in the same setting. Additionally, the result for Kannan contraction mappings is applied to obtain the solution of an integral equation.

Keywords and Phrases: Mann iteration sequence, convex G_b-metric spaces, fixed points, integral equation

2020 Mathematics Subject Classification : 47H10, 47H09.

Introduction

Mann iteration scheme which is widely used for establishing the existence of, and approximating fixed points in abstract spaces was introduced by Mann (1953). However, its convergence to existing fixed points depends on the convexity of the ambient space. Hence to apply the Mann iteration scheme in any space, it is necessary that the space satisfies a defined condition of convexity that imparts a convex structure to the space.

For metric spaces, one such condition of convexity is that defined by Takahashi(1970), and a metric space satisfying it is said to be convex in the sense of Takahashi. Many researchers have proved fixed point results in this space. For instance, Eke et al. (2018) proved common fixed point theorems for nonself contraction mappings in convex metric spaces. Also this space has been generalized to convex b-metric space in Chen et al. (2020), the authors proved some fixed point theorems for Banach contraction

mappings and Kannan contraction mappings in convex b-metric spaces using the Mann iteration scheme.
The notion of metric spaces has been generalized to the notion of G-metric spaces by Mustafa and Sims (2006) which in turn has been extended to the notion of G_b -metric spaces in Aghajani et al. (2014). Accordingly, Takahashi's condition of convexity, which was originally defined for metric spaces, has been formulated for G_b -metric spaces by Ji et al. (2023).

The following question is pertinent: for G_b -metric spaces that are convex in the sense of Takahashi, under what conditions will the Mann iteration scheme converge to fixed points of a particular class of contraction mappings defined therein? This question has already been answered for Banach contraction mappings introduced by Banach (1922) in the work of Ji et al. (2023); and this is a motivation for finding respective answers for Kannan (1968) contraction mappings, Chatterjea (1972) contraction mappings and Hardy-Rogers (11973) contraction mappings. This research will provide answer to the underlining question and equally apply the obtained result to find the solution of an integral equation.

Preliminary
The brief discussions of metric spaces and its generalizations will be stated here as some of them are paramount to our main results.

Definition 2.1 Frechet (1906) : Let X be a non-empty set, and $d: X \times X \to [0,\infty)$ be a mapping. For every $x,y,z \in X$, if d satisfies the following conditions:
(i) $d(x,y) = 0$ if and only if $x = y$;
(ii) $d(x,y) = d(y,x)$;
(iii) $d(x,y) \le d(x,z) + d(y,z)$.
Then d is called a metric on X and the pair (X, d) is called a metric space.

Czwerik (1993) generalized the notion of metric space by adding a constant $s \ge 1$ to the triangle inequality.

Definition 2.2 : Let X be a non-empty set, and $d_b : X \times X \to [0,\infty)$ be a mapping. For every $x,y,z \in X$, if d satisfies the following conditions:
(i) $d_b(x,y) = 0$ if and only if $x = y$;
(ii) $d_b(x,y) = d_b(y,x)$;
(iii) There exists a real number $s \ge 1$ such that $d_b(x,y) \le s[d_b(x,z) + d_b(y,z)]$,
then d_b is called a b-metric on X and the pair (X, d_b) is called a b-metric space.

In 1970, Takahashi introduced a convex structure to the axioms of a metric space as follows.

Definition 2.3 : Let (X, d) be a metric space and I=[0, 1]. A continuous function $w : X \times X \times [0, 1] \to X$ is said to be a convex structure on X if for each $x,y,u \in X$ and $\alpha \in I$ $d(u,w(x,y;\alpha)) \le \alpha d(u,x) + (1-\alpha) d(u,y)$ holds. A metric space (X, d) with a convex structure w is called a convex metric space.

In 2014, Aghajani et al. (2014) introduced the notion of G_b - metric spaces which are another generalization of metric spaces.

Definition 2.4 : Let X be a nonempty set. Suppose that a mapping $G_b : X \times X \times X \to [0,\infty)$ satisfies the following conditions:
(1) $G_b(x, y, z) = 0$ if $x = y = z$;
(2) $0 < G_b(x, x, y)$ for all $x, y \in X$ with $x \neq y$;
(3) $G_b(x, x, y) \leq G_b(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$;
(4) $G_b(x, y, z) = G_b(x, z, y) = G_b(z, x, y) = \cdots$ (symmetry in three variables);
(5) there exists a real number $s \geq 1$ such that
$G_b(x, y, z) \leq s[G_b(x, u, u) + G_b(u, y, z)]$ for all $x, y, z \in X$.
Then G_b is called a G_b - metric on X and the pair (X, G_b) is called G_b metric spaces.

Example 2.5 : Let $X = [0,\infty)$ and $G_b : X \times X \times X \to [0,\infty)$ be a mapping defined by

$$G_b(x,y,z) = \left[\frac{1}{3}\left(|x-y| + |y-z| + |z-x|\right)\right]^2$$

Then G_b is called a G_b -metric on X.

The following definitions and motivations are found in Ji et al. (2023).

Definition 2.6 : Let (X, G_b) be a G_b -metric space with coefficient $s \geq 1$ and I = [0,1]. A mapping $w : X \times X \times [0, 1] \to X$ is called a convex structure on X if for $x, y, u, v \in X$ and $\alpha \in I$, the inequality
$G_b(u, v, w(x, y; \alpha)) \leq \alpha G_b(u, v, x) + (1-\alpha) G_b(u, v, y)$ holds.
If so, the triplet (X, G_b, w) is called a convex G_b -metric space.

Definition 2.7 : Let (X, G_b, w) be a convex G_b -metric space and $T: X \to X$ be a mapping. The generalized Mann iteration scheme is;
$x_{n+1} = w(x_n, Tx_n; \alpha_n)$, $n \in N$,
where $x_n \in X$ and $\alpha_n \in [0, 1]$. The sequence $\{x_n\}$ is called Mann iteration sequence for T.

Theorem 2.8 : Let (X, G_b, w) be a complete convex G_b - metric space with constant $s \geq 1$ and $T: X \to X$ be a mapping such that
$G_b(Tx, Ty, Tz) \leq \lambda G_b(x, y, z)$
for all $x, y, z \in X$ and $\lambda \in [0, 1)$. Suppose that the sequence $\{x_n\}$ is generated by the Mann iterative scheme and for $x_0 \in X$. If the sequence $\{\alpha_n\} \in (0, 1)$ converges to $\alpha < \frac{1 - s^2\lambda}{s^2 - s^2\lambda}$ and $\lambda < \frac{1}{s^2}$, then T has a unique fixed point. $x^* \in X$ Moreover, T is G- continuous at x^*.

The following contractive mappings are employed in our research.

Definition 2.9 Kannan (1968)**:** Let (X, d) be a metric space and $T: X \to X$ a self mapping. T is said to be Kannan contraction mappings if there exists $\alpha \in \left(0, \frac{1}{2}\right)$ such that for all $x, y \in X$ the following inequality holds

$d(Tx, Ty) \leq \alpha \left[d(x, Tx) + d(y, Ty)\right].$

Definition 2.10 Chatterjea (1972): Let (X, d) be a metric space and $T: X \to X$ a self mapping. T is said to be Chatterjea contraction mappings if there exists $\beta \in \left(0, \frac{1}{2}\right)$ such that for all $x, y \in X$ the following inequality holds

$d(Tx, Ty) \leq \beta\left[d(x, Ty) + d(y, Tx)\right].$

Definition 2.11 Hardy and Rogers (1973): Let (X, d) be a metric space and $T: X \to X$ a self mapping. T is said to be Hardy and Rogers contraction mappings if there exists $a, b, c, e, f \in (0, 1)$ with a + b + c + e + f<1 such that for all $x, y \in X$ the following inequality holds,

$d(Tx, Ty) \leq ad(x, y) + bd(x, Tx) + cd(y, Ty) + ed(x, Ty) + fd(y, Tx).$

Fixed Point Results

Fixed point theorems that establish the conditions under which the Mann iteration sequence converges for Chatterjea contraction mappings, Hardy and Rogers contraction mappings in convex G_b -metric spaces are proved in this section with examples illustrating our results.

Theorem 3.3: Let (X, G_b,w) be a complete convex G_b -metric space. Suppose $\{\alpha_n\}$ is a sequence in (0,1) that converges to $\alpha < \frac{1-s^2\beta(1+s)}{s(s+s\beta)}$, where $\beta \in \left[0, \frac{1}{s^2(s+1)}\right)$, then the Mann iteration sequence converges to a unique fixed point of the mapping $T: X \to X$ satisfying the condition:

$G_b(Tx,Ty,Ty) \leq \beta[G_b(x,Ty,Ty) + G_b(y,Tx,Tx)]$ for all $x, y \in X$.

Proof:

For ease, G shall be used for G_b.

For any $n \in N$,

$$\begin{aligned} G(x_n,x_n,x_{n+1}) &= G(x_n,x_n,W(x_n,Tx_n;\alpha_n)) \\ &\leq \alpha_n G(x_n,x_n,x_n) + (1-\alpha_n)G(x_n,x_n,Tx_n) \qquad (3.1) \\ &= (1-\alpha_n)G(x_n,x_n,Tx_n) \end{aligned}$$

$$
\begin{aligned}
G\left(x_n,x_n,Tx_n\right) &= G\left(x_n,Tx_n,Tx_n\right) \\
&\le s\left[G\left(x_n,Tx_{n-1},Tx_{n-1}\right)+G\left(Tx_{n-1},Tx_n,Tx_n\right)\right] \\
&= s[G\left(w\left(x_{n-1},Tx_{n-1},\alpha_{n-1}\right),Tx_{n-1},Tx_{n-1}\right) + G(Tx_{n-1},Tx_n,Tx_n)] \\
&\le s[\alpha_{n-1}G\left(x_{n-1},Tx_{n-1},Tx_{n-1}\right)+ \left(1-\alpha_{n-1}\right)G\left(Tx_{n-1},Tx_{n-1},Tx_{n-1}\right) \\
&\quad + \beta\left(G\left(x_{n-1},Tx_n,Tx_n\right) + G\left(x_n,Tx_{n-1},Tx_{n-1}\right)\right)] \\
&= s\alpha_{n-1}G\left(x_{n-1},x_{n-1},Tx_{n-1}\right) + s\beta G(x_{n-1},Tx_n,Tx_n) \\
&\quad + s\beta G\left(x_n,Tx_{n-1},Tx_{n-1}\right).
\end{aligned} \tag{3.2}
$$

$$
\begin{aligned}
G\left(x_{n-1},Tx_n,Tx_n\right) &= G\left(x_{n-1},x_{n-1},Tx_n\right) = G(Tx_n,x_{n-1},x_{n-1}) \\
&\le s\left[G\left(Tx_n,x_n,x_n\right)+ G\left(x_n,x_{n-1},x_{n-1}\right)\right] \\
&\le s[G\left(x_n,x_n,Tx_n\right) +\left(1-\alpha_{n-1}\right) G(x_{n-1},x_{n-1},Tx_{n-1})] \\
&= sG\left(x_n,x_n,Tx_n\right) + s\left(1-\alpha_{n-1}\right)G\left(x_{n-1},x_{n-1},Tx_{n-1}\right).
\end{aligned} \tag{3.3}
$$

$$
\begin{aligned}
G\left(x_n,Tx_{n-1},Tx_{n-1}\right) &= G(w(x_{n-1},Tx_{n-1};\ \alpha_{n-1}),Tx_{n-1},Tx_{n-1}) \\
&\le \alpha_{n-1}G\left(x_{n-1},x_{n-1},Tx_{n-1}\right)
\end{aligned} \tag{3.4}
$$

Substituting (3.3)and (3.4) into (3.2) yields,

$$
\begin{aligned}
G\left(x_n,x_n,Tx_n\right) &\le s\alpha_{n-1}G(x_{n-1},x_{n-1},Tx_{n-1}) + s^2\beta G\left(x_n,x_n,Tx_n\right) \\
&\quad + s^2\beta\left(1-\alpha_{n-1}\right)G\left(x_{n-1},x_{n-1},Tx_{n-1}\right)+s\beta\alpha_{n-1}G\left(x_{n-1},x_{n-1},Tx_{n-1}\right).
\end{aligned}
$$

This implies

$$
\begin{aligned}
\left(1-s^2\beta\right) G\left(x_n,x_n,Tx_n\right) &\le (s\alpha_{n-1} + s^2\beta\left(1-\alpha_{n-1}\right)+s\beta\alpha_{n-1})G(x_{n-1},x_{n-1},Tx_n) \\
&\le \left(s\alpha_{n-1} + s^2\beta + s\beta\alpha_{n-1}\right)G(x_{n-1},x_{n-1},Tx_{n-1}).
\end{aligned}
$$

This gives,

$$
G\left(x_n,x_n,Tx_n\right) \le \frac{s\alpha_{n-1} + s^2\beta + s\beta\alpha_{n-1}}{1-s^2\beta}G\left(x_{n-1},x_{n-1},Tx_{n-1}\right). \tag{3.5}
$$

Let $k_n = \dfrac{s\alpha_{n-1} + s^2\beta + s\beta\alpha_{n-1}}{1-s^2\beta}$, α_n converges to α and $\lim_{n\to\infty} k_n = k.$

Then $k = \dfrac{s\alpha + s^2\beta + s\beta\alpha}{1-s^2\beta} < \dfrac{1}{s},$

Hence (3.5) becomes

$$
\begin{aligned}
G\left(x_n,x_n,Tx_n\right) &\le k_{n-1}G\left(x_{n-1},x_{n-1},Tx_{n-1}\right) \\
&\le k_{n-1} \times k_{n-2}G\left(x_{n-2},Tx_{n-2},Tx_{n-2}\right)
\end{aligned}
$$

$$\begin{aligned}&\le\\&\vdots\\&\le \textstyle\prod_{i=0}^{n-1} k_i G(x_0,x_0,Tx_0)\end{aligned} \tag{3.5a}$$

$$\begin{aligned}G(x_n,x_n,x_{n+1}) &\le (1-\alpha_{n-1})G(x_n,x_n,Tx_n)\\&\le G(x_n,x_n,Tx_n)\\&\le \textstyle\prod_{i=0}^{n-1} k_i G(x_0,x_0,Tx_0).\end{aligned} \tag{3.5b}$$

For $p \ge 1$ and $n \in N$, repeated application of the rectangle inequality property gives:

$$\begin{aligned}G(x_n,x_n,x_{n+P}) &\le sG(x_n,x_{n+1},x_{n+1}) + sG(x_{n+1}+x_{n+P}+x_{n+P})\\&\le sG(x_n,x_{n+1},x_{n+1}) + s^2G(x_{n+1},x_{n+2},x_{n+2}) + \cdots + s^pG(x_{n+p-1},x_{n+p},x_{n+p})\\&\le s\textstyle\prod_{i=0}^{n-1} k_iG(x_0,x_0,Tx_0) + s^2\prod_{i=0}^{n} k_iG(x_0,x_0,Tx_0) + \cdots + s^p\prod_{i=0}^{n+p-2} k_iG(x_0,x_0,Tx_0)\\&= (sU_{n-1} + s^2U_n + s^3U_{n+1} + \cdots + s^pU_{n+p-2})G(x_0,x_0,Tx_0)\end{aligned}$$

where $U_{n-1} = \prod_{i=0}^{n-1} k_i$. $\lim_{n\to\infty} \dfrac{U_n}{U_{n-1}} = \lim_{n\to\infty} K_n = k < \dfrac{1}{s} \le 1$

and so by ratio test, $\sum U_{n-1}$ is convergent.

According to the theorem which states that if a sequence is convergent, the sequence of its terms converges to 0 then $\lim_{n\to\infty} U_{n-1} = 0$. This implies,

$U_{n-1} = U_n = \cdots = U_{n+p-2} = 0$ as $n \to \infty$, therefore, $G(x_n,x_n,x_{n+p}) \to 0$ as $n \to \infty$.

This implies that $\{x_n\}$ is a Cauchy sequence and converges to a point x^* in X, since the space is complete; and from (3.5b)

$$\lim_{n\to\infty} G(x_n,x_n,Tx_n) = 0 \tag{3.5c}$$

Next, the existence of the fixed point of the mapping is established.

$$\begin{aligned}G(x^*,Tx^*,Tx^*) &\le s\left[G(x^*,x_n,x_n) + G(x_n,Tx^*,Tx^*)\right]\\&\le sG(x^*,x_n,x_n) + s^2\left[G(x_n,Tx_n,Tx_n) + G(Tx_n,Tx^*,Tx^*)\right]\\&\le sG(x^*,x_n,x_n) + s^2G(x_n,Tx_n,Tx_n) + s^2\beta G(x_n,Tx^*,Tx^*) + s^2\beta G(x^*,Tx_n,Tx_n)\\&\le sG(x^*,x_n,x_n) + s^2G(x_n,x_n,Tx_n) + s^2\beta G(x_n,Tx^*,Tx^*)\\&\quad + s^3\beta G(x^*,x_n,x_n) + s^3\beta G(x_n,Tx_n,Tx_n)\end{aligned}$$

As $n \to \infty$, then from the facts that $x_n \to x$ and (3.5c), it follows that:

$G(x^*,Tx^*,Tx^*) \le s^2\beta G(x^*,Tx^*,Tx*)$. This implies $(1-s^2\beta)G(x^*,Tx^*,Tx^*) \le 0$.

Since $1-s^2\beta > 0$, $G(x^*,Tx^*,Tx^*) \le 0$. But $G(x^*,Tx^*,Tx^*) \ge 0$.

Therefore $G\left(x^*, Tx^*, Tx^*\right) = 0$. Hence x^* is a fixed point of T.

Suppose there exists $y^* \neq x^*$ such that $y^* \in X$ and $Ty^* = y^*$. $y^* \neq x^*$ implies that $G\left(x^*, y^*, y^*\right) > 0$. By property,

$$\begin{aligned} G\left(x^*, y^*, y^*\right) &= G\left(Tx^*, Ty^*, Ty^*\right) \\ &\leq \beta\left[G\left(x^*, Ty^*, Ty^*\right) + G\left(y^*, Tx^*, Tx^*\right)\right] \\ &= \beta\left[G\left(Tx^*, Ty^*, Ty^*\right) + G\left(Ty^*, Tx^*, Tx^*\right)\right] \\ &= 2\beta G\left(Tx^*, Ty^*, Ty^*\right) \\ &= 2\beta G\left(x^*, y^*, y^*\right) \end{aligned}$$

This implies that $(1-2\beta)G\left(x^*, y^*, y^*\right) \leq 0$ but $1-2\beta > 0$. Therefore $G\left(x^*, y^*, y^*\right) \leq 0$, a contradiction to the fact that $G\left(x^*, y^*, y^*\right) > 0$. Hence, x^* is a unique fixed point of T.

The following example illustrates the above theorem.

Example : Let $X = [0, \infty)$ and $Tx = \frac{x}{2}$ for $x \in X$. For any $x, y, z \in X$, we define $G_b : X \times X \times X \to [0, \infty)$ with the formula $G_b(x, y, z) = |x-y| + |y-z| + |x-z|$. Thus (X, G_b) is a G_b-metric space with constant s= 1. Also, x = 0 is the only fixed point of T in X. Consider the mapping $w(x, y; \alpha) = \alpha x + (1-\alpha)y$ we prove that w is a convex structure on (X, G_b) by showing that
$G_b\left(x, y, w(u, v; \alpha)\right) \leq \alpha G_b(x, y, u) + (1-\alpha)G_b(x, y, v)$.

$$\begin{aligned} G_b(x, y, w(u, v; \alpha)) &= |x-y| + |x-w| + |y-w| \\ &= |x-w| + |x-(\alpha x + (1-\alpha)y| + \left|y - \left(\alpha x + (1-\alpha)y\right)\right| \\ &= (\alpha+1)\ |x-y| + (1-\alpha)|x-y| \\ &= (\alpha+1+1-\alpha)\ |x-y| = 2|x-y| \end{aligned}$$

$$\begin{aligned} \alpha G_b(x, y, u) + (1-\alpha)\, G_b(x, y, v) &= |x-y| + \alpha\ |y-u| \\ &\geq \alpha|x-u| + (1-\alpha)|y-v| + (1-\alpha)\ |x-v| \\ &= |x-u| + \alpha(|x-y| + |u-y| + (1-\alpha)\ (|x-u| + |u-y|) \\ &\quad + (1-\alpha)\ (|x-v| + |v-y|) \\ &\geq |x-y| + \alpha|x-y| + (1-\alpha)|x-y| \\ &= 2|x-y| = G_b\left(x, y, W(u, v; \alpha)\right) \end{aligned}$$

Hence w is a convex structure on $\left(X,G_b\right)$ and so $\left(X,G_b,w\right)$ is a convex G_b-metric space with s=1.

$$G_b\left(x,y,y\right) = 2\left|x-y\right|$$

$$G_b\left(x,\ Ty,\ Ty\right)=2\ \left|\frac{2x-y}{2}\right|$$

$$G_b\left(y,Tx,Tx\right)=\ 2\ \left|\ \frac{2y-x}{2}\right|$$

$$\begin{aligned}G_b\left(Tx,Ty,Ty\right) &= 2\left|Tx-Ty\right| \\ &= 2\ \left|\frac{x}{2}-\frac{y}{2}\right| \ = 2\times\frac{1}{3}\ \left|\frac{3x-3y}{2}\right| \\ &= \frac{2}{3}\ \left|\ \frac{2x-y}{2}+\ \frac{x-2y}{2}\ \right| \\ &\le \frac{2}{3}\left(\ \left|\frac{2x-y}{2}\right| + \left|\frac{2y-x}{2}\right|\right) \\ &= \frac{2}{3}(G\left(x,Ty,Ty\right)\ +\ G\left(y,Tx,Tx\right)\end{aligned}$$

Hence T $= \dfrac{x}{2}$ is a Chatterjea contraction on $\left(X,G_b,w\right)$ with $\beta=\dfrac{1}{3}<\dfrac{1}{2}=\dfrac{1}{s^2(s+1)}$.

Let $\alpha_n=\dfrac{1}{5}=\dfrac{1-s^2\beta(s+1)}{s(s+s\beta)}$, then $\alpha=\lim_{n\to\infty}\alpha_n=\dfrac{1}{5}$.

$$\begin{aligned}x_n &= w\left(x_{n-1},Tx_{n-1};\alpha_{n-1}\right) \\ &= \frac{1}{5}x_{n-1}+\frac{4}{5}Tx_{n-1} \\ &= \frac{1}{5}x_{n-1}+\frac{4}{10}x_{n-1} \\ &= \left(\frac{3}{5}\right)x_{n-1} \\ &= \left(\frac{3}{5}\right)^2 x_{n-2} \\ &= \left(\frac{3}{5}\right)^3 x_{n-3} \\ &\vdots \\ &= \left(\frac{3}{5}\right)^n x_0\end{aligned}$$

Since $Tx=\dfrac{x}{2}$ then $Tx_n=\dfrac{x_n}{2}$. As $n\to\infty,\ x_n\to 0,\ Tx_n\to 0$ and the result follows.

Next is the result for Hardy-Rogers contraction mappings.

Theorem 3.5: Let (X,G_b,w) be a complete convex G_b-metric space. Suppose $\{\alpha_n\}$ is a sequence in (0,1) that converges to $\alpha < \frac{1-s^3(a++c+2d+e)}{s^2}$, where $a+b+c+2d+e < \frac{1}{s^3}$, then the Mann iteration sequence converges to a unique fixed point of the mapping $T : X \to X$ satisfying the condition:

$$G_b(Tx,Ty,Ty) \le aG_b(x,y,y) + bG_b(x,Tx,Tx) + cG_b(y,Ty,Ty) + dG_b(x,Ty,Ty) + eG_b(y,Tx,Tx)$$

for all $x,y \in X$.

Proof: For ease, G shall be used for G_b.

$$\begin{aligned} G(x_n,x_n,x_{n+1}) &= G(x_n,x_n,W(x_n,Tx_n;\alpha_n)) \\ &\le \alpha_n G(x_n,x_n,x_n) + (1-\alpha_n)G(x_n,x_n,Tx_n) \\ &= (1-\alpha_n)G(x_n,x_n,Tx_n) \end{aligned} \tag{3.6}$$

For any $n \in N$,

$$\begin{aligned} G(x_n,x_n,Tx_n) &= G(x_n,Tx_n,Tx_n) \\ &\le s\left[G(x_n,Tx_{n-1},Tx_{n-1}) + G(Tx_{n-1},Tx_n,Tx_n)\right] \\ &= s[G(w(x_{n-1},Tx_{n-1},\alpha_{n-1}),Tx_{n-1},Tx_{n-1}) + G(Tx_{n-1},Tx_n,Tx_n)] \\ &\le s[\alpha_{n-1}G(x_{n-1},Tx_{n-1},Tx_{n-1}) + aG(x_{n-1},x_n,x_n) \\ &\quad + bG(x_{n-1},Tx_{n-1},Tx_{n-1}) + cG(x_n,Tx_n,Tx_n) \\ &\quad + dG(x_{n-1},Tx_n,Tx_n) + eG(x_n,Tx_{n-1},Tx_{n-1})] \end{aligned} \tag{3.7}$$

$$\begin{aligned} G(x_{n-1},x_n,x_n) &= G(x_n,x_{n-1},x_{n-1}) \\ &= G(w(x_{n-1},Tx_{n-1};\alpha_{n-1}),Tx_{n-1},Tx_{n-1}) \\ &\le \alpha_{n-1}G(x_{n-1},Tx_{n-1},Tx_{n-1}) \end{aligned} \tag{3.8}$$

$$\begin{aligned} G(x_{n-1},Tx_n,Tx_n) &= G(x_{n-1},x_{n-1},Tx_n) = G(Tx_n,x_{n-1},x_{n-1}) \\ &\le s\left[G(Tx_n,x_n,x_n) + G(x_n,x_{n-1},x_{n-1})\right] \\ &\le s[G(x_n,x_n,Tx_n) + (1-\alpha_{n-1})G(x_{n-1},x_{n-1},Tx_{n-1})] \\ &= sG(x_n,x_n,Tx_n) + s(1-\alpha_{n-1})G(x_{n-1},x_{n-1},Tx_{n-1}) \end{aligned} \tag{3.9}$$

$$\begin{aligned} G(x_n,Tx_{n-1},Tx_{n-1}) &= G(w(x_{n-1},Tx_{n-1};\alpha_{n-1}),Tx_{n-1},Tx_{n-1}) \\ &\le \alpha_{n-1}G(x_{n-1},x_{n-1},Tx_{n-1}) \end{aligned} \tag{3.10}$$

Substituting (3.8), (3.9), and (3.10) into (3.7) yields,

$$\begin{aligned} G(x_n,x_n,Tx_n) &\le s[\alpha_{n-1}G(x_{n-1},Tx_{n-1},Tx_{n-1}) + a(1-\alpha_{n-1})G(x_{n-1},Tx_{n-1},Tx_{n-1}) \\ &\quad + bG(x_{n-1},Tx_{n-1},Tx_{n-1}) + cG(x_n,x_n,Tx_n) + sdG(x_n,x_n,Tx_n) \\ &\quad + sd(1-\alpha_{n-1}G(x_{n-1},Tx_{n-1},Tx_{n-1}) + e\alpha_{n-1}G(x_{n-1},Tx_{n-1},Tx_{n-1})] \end{aligned}$$

$$\begin{aligned}
&= s\alpha_{n-1}G(x_{n-1},Tx_{n-1},Tx_{n-1}) + sa(1-\alpha_{n-1})G(x_{n-1},Tx_{n-1},Tx_{n-1}) \\
&+ sbG(x_{n-1},Tx_{n-1},Tx_{n-1}) \\
&+ scG(x_n,x_n,Tx_n) + s^2dG(x_n,x_n,Tx_n) \\
&+ s^2d\,G(x_{n-1},Tx_{n-1},Tx_{n-1}) + se\alpha_{n-1}G(x_{n-1},Tx_{n-1},Tx_{n-1}).
\end{aligned}$$

In view of $0<\alpha<1$ which implies that $0<1-\alpha<1$, $s\geq 1$, and the last inequality, we have

$$\begin{aligned}
G(x_n,x_n,Tx_n) &\leq s\alpha_{n-1}G(x_{n-1},Tx_{n-1},Tx_{n-1}) + s^2a\,G(x_{n-1},Tx_{n-1},Tx_{n-1}) \\
&+ s^2bG(x_{n-1},Tx_{n-1},Tx_{n-1}) + s^3cG(x_n,x_n,Tx_n) + s^3dG(x_n,x_n,Tx_n) \\
&+ s^2d\,G(x_{n-1},Tx_{n-1},Tx_{n-1}) + se^2\alpha_{n-1}G(x_{n-1},Tx_{n-1},Tx_{n-1})
\end{aligned}$$

$$(1-s^3(c+d))G(x_n,x_n,Tx_n) \leq (s\alpha_{n-1} + s^2a + s^2b + s^2d + s^2e\alpha_{n-1})G(x_{n-1},Tx_{n-1},Tx_{n-1})$$

$$G(x_n,x_n,Tx_n) \leq \frac{s\alpha_{n-1} + s^2(a+b+d+e\alpha_{n-1}}{1-s^3(c+d)}\,G(x_{n-1},Tx_{n-1},Tx_{n-1}) \tag{3.11}$$

Let $k_{n-1} = \dfrac{s\alpha_{n-1} + s^2(a+b+d+e\alpha_{n-1})}{1-s^3(c+d)}$ and $\lim_{n\to\infty}k_{n-1} = k$, then $k = \dfrac{s\alpha + s^2(a+b+c+d+e\alpha)}{1-s^3(c+d)} < \dfrac{1}{s}$.

So (3.11) becomes

$$\begin{aligned}
G(x_n,x_n,Tx_n) &\leq k_{n-1}G(x_{n-1},x_{n-1},Tx_{n-1}) \\
&\leq k_{n-1}\cdots k_{n-2}G(x_{n-2},Tx_{n-2},Tx_{n-2}) \\
&\leq \\
&\vdots \\
&\leq \textstyle\prod_{i=0}^{n-1}k_iG(x_0,x_0,Tx_0).
\end{aligned} \tag{3.12}$$

$$\begin{aligned}
G(x_n,x_n,x_{n+1}) &\leq (1-\alpha_{n-1})G(x_n,x_n,Tx_n) \\
&\leq G(x_n,x_n,Tx_n) \\
&\leq \textstyle\prod_{i=0}^{n-1}k_iG(x_0,x_0,Tx_0) \\
&\leq \textstyle\prod_{i=0}^{n}k_iG(x_0,x_0,Tx_0).
\end{aligned}$$

For $n\in N$ and $p\geq 1$ and applying the rectangle inequality property repeatedly, it follows that

$$G(x_n,x_n,x_{n+p}) \leq s\textstyle\prod_{i=0}^{n}k_iG(x_0,x_0,Tx_0) + s^2\prod_{i=0}^{n+1}k_iG(x_0,x_0,Tx_0) + \cdots + s^p\prod_{i=0}^{n+p-1}k_iG(x_0,x_0,Tx_0)$$

Let $z_{n+i} = s^{i+1}\prod_{i=0}^{n+1}k_i$, $i = 0,1,\cdots,p-1$ then

$$G(x_n,x_n,x_{n+p}) \leq (z_n + z_{n+1} + \cdots + z_{n+p-1})\,G(x_0,x_0,Tx_0),$$

$$\lim_{i\to\infty}\frac{z_{n+i+1}}{z_{n+i}} = \lim_{i\to\infty}\frac{s^{i+2}\prod_{i=0}^{n+i+1}k_i}{s^{i+1}\prod_{i=0}^{n+1}k_i}$$

$$= \lim_{i\to\infty} sk_{n+i+1} < s\left(\frac{1}{s}\right) = 1$$

so by the Ratio Test , $\sum_{i=0}^{\infty} z_{n+i}$ is convergent and hence $\lim_{n\to\infty} \sum_{i=n}^{n+p-1} z_n = 0$.

This implies that $\lim_{n\to\infty} G\left(x_n, x_n, x_{n+p}\right) = 0$.

.

Thus $\{x_n\}$ is a Cauchy sequence and since $\left(X, G_b, w\right)$ is complete, this implies that $x_n \to x^* \in X$.

Also, let $u_n = \prod_{i=0}^{n} k_i$ then, $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} k_{n+1} < \frac{1}{s} < 1$, and by the Ratio Test, $\sum_{n=0}^{\infty} u_n$ is convergent which implies that $u_n \to 0$ as $n \to \infty$, that is, $\prod_{i=0}^{n} k_i \to 0$ as $n \to \infty$ and from (3.12)

$$\lim_{n\to\infty} G\left(x_n, x_n, Tx_n\right) = 0 \qquad (3.13)$$

Next the existence of the fixed point of the mapping is established

$$\begin{aligned} G\left(x^*, Tx^*, Tx^*\right) &\le s[G\left(x^*, x_n, x_n\right) + G\left(x_n, Tx^*, Tx^*\right)] \\ &\le s[G\left(x^*, x_n, x_n\right) + s\left(G\left(x_n, Tx_n, Tx_n\right) + G\left(Tx_n, Tx^*, Tx^*\right)\right)] \end{aligned} \qquad (3.14)$$

$$\begin{aligned} G\left(Tx^*, Tx^*, Tx^*\right) &\le aG\left(x_n, x^*, x^*\right) + bG\left(x_n, Tx_n, Tx_n\right) + cG\left(x^*, Tx^*, Tx^*\right) \\ &\quad + dG\left(x_n, Tx^*, Tx^*\right) + eG\left(x^*, Tx_n, Tx_n\right) \\ &\le aG\left(x_n, x^*, x^*\right) + bG\left(x_n, Tx_n, Tx_n\right) + cG\left(x^*, Tx^*, Tx^*\right) \\ &\quad + dG\left(x_n, Tx^*, Tx^*\right) + e\left[sG\left(x^*, x_n, x_n\right) + G\left(x_n, Tx_n, Tx_n\right)\right] \end{aligned} \qquad (3.15)$$

Substituting (3.15) into (3.14) yields

$$\begin{aligned} G\left(x^*, Tx^*, Tx^*\right) &\le sG\left(x^*, x_n, x_n\right) + s^2 G\left(x_n, Tx_n, Tx_n\right) \\ &\quad + s^2 aG\left(x_n, x^*, x^*\right) + s^2 bG\left(x_n, Tx_n, Tx_n\right) + s^2 cG\left(x^*, Tx^*, Tx^*\right) \\ &\quad + s^2 dG\left(x_n, Tx^*, Tx^*\right) + s^3 eG\left(x^*, x_n, x_n\right) + s^3 eG\left(x_n, Tx_n, Tx_n\right). \end{aligned}$$

Taking limit as $n \to \infty$ yields

$$G\left(x^*, Tx^*, Tx^*\right) \le s^2 cG\left(x^*, Tx^*, Tx^*\right) + s^2 dG\left(x^*, Tx^*, Tx^*\right)$$

and this implies that $\left(1 - s^2\left(c+d\right)\right)G\left(x^*, Tx^*, Tx^*\right) \le 0$. But $c + d < \frac{1}{s^3}$ which implies that $s^2\left(c+d\right) < \frac{1}{s} < 1$ which implies that $1 - s^2\left(c+d\right) > 0$, therefore $G\left(x^*, Tx^*, Tx^*\right) \le 0$, and since $G\left(x^*, Tx^*, Tx^*\right) \ge 0$, it implies that $G\left(x^*, Tx^*, Tx^*\right) = 0$ and so $Tx^* = x^*$. Hence, x^* is a fixed point of T.

Suppose there exists $y^* \in X$ such that $y^* \neq x^*$ and $Ty^* = y^*$. Then $G(x^*, y^*, y^*) > 0$.

$$\begin{aligned} G(x^*, y^*, y^*) &= G(Tx^*, Ty^*, Ty^*) \\ &\leq aG(x^*, y^*, y^*) + bG(x^*, Tx^*, Tx^*) + cG(y^*, Ty^*, Ty^*) \\ &+ dG(x^*, Ty^*, Ty^*) + eG(y^*, Tx^*, Tx^*) \\ &= aG(x^*, y^*, y^*) + dG(x^*, Ty^*, Ty^*) + eG(y^*, Tx^*, Tx^*) \\ &(1-(a+d+e))\, G(x^*, y^*, y^*) \leq 0. \end{aligned}$$

Since $1-(a+d+e) > 0$, it implies that $G(x^*, y^*, y^*) \leq 0$, a contradiction. Hence x^* is a unique fixed point of T.

The following example illustrates this theorem.

Example 3.6 : Let $X = [0,\infty)$ and $T: X \to X$ be defined as $T = \frac{x}{3}$ for all $x \in X$ For any $x, y, z \in X$, let $G_b : X \times X \times X \to [0,\infty)$ defined by $G_b(x, y, z) = \frac{1}{3}(|x-y| + |y-z| + |x-z|)^2$ be a G_b-metric on X. The mapping defined by $w(x, y; \alpha) = \alpha x + (1-\alpha)y$ is a convex structure on the space. Accordingly, (X, G_b,w) is a convex G_b-metric with s = 2. All the hypothesis of theorem 4.2.3 are satisfied and 0 is a unique fixed point of T.

The following Corollary is obtained by assigning $a = d = e = 0$ in Theorem 3.5.

Corollary 3.7 : Let (X, G_b,w) be a complete convex G_b-metric space. Suppose $\{\alpha_n\}$ is a sequence in (0,1) that converges to $\alpha < \frac{1-s\beta(s+1)}{s}$,where $\beta \in \left[0, \frac{1}{s(s+1)}\right]$, then the Mann iteration sequence $\{x_n\}$ converges to a unique fixed point of the mapping $T : X \to X$ satisfying the condition:
$G_b(Tx, Ty, Ty) \leq \beta[G_b(x, Tx, Tx) + G_b(y, Ty, Ty)]$ for all $x, y \in X$.

Remarks 3.8: (i) Theorem 3.5 is a generalization of the result of Ji et al. (2023) because $b = c = d = e = 0$ in Hardy and Rogers contraction mappings gives the result of Ji et al.(2023).
(ii) Theorem 3.5 is a generalization of Theorem 1 and Theorem 2 of Chen et al.(2020), the authors proved existence of the unique fixed points for Banach contraction mappings and Kannan contraction mappings in the framework of convex b-metric spaces utilizing the Mann iteration algorithm. In Theorem 3.5, our spaces is more general compare to that of Chen et al. (2020) likewise our mappings.

Application

In this section, Corollary 3.7 is applied to obtain the existence theorem of an integral equation. Consider the integral equations below,

$$x(t) = f(t) + \gamma\int_a^b u(t,\tau)K_1(\tau, x(\tau))d\tau \int_a^b u(t,\tau)K_2(\tau, x(\tau))d\tau \qquad (4.1)$$

for $t \in [a,b]$, where $f: [a,b] \to R$, $u: [a,b] \times [a,b] \to R$ and $K_1, K_2 : [a,b] \to R$ are all continuous functions on [a,b]. Denoted by $X = (C([a,b], R)$ is the space of all continuous functions on [a,b]. Endowed with this space is the G_b-metric

$$G_b(x,y,z) = \left(\sup_{t\in[a,b]}|x(t)-y(t)| + \sup_{t\in[a,b]}|y(t)-z(t)| \sup_{t\in[a,b]}|z(t)-x(t)|\right)^2.$$

The function $w : X \times X \times (0,1)$ defined as $w(x,y;\alpha) = \alpha x + (1-\alpha)y$ is a convex structure on the space. Accordingly, (X, G_b, w) is a complete convex G_b-metric space with s =2; consider $T: X \to X$ defined by ,

$$Tx(t) = f(t) + \gamma\int_a^b u(t,\tau)K_1(\tau, x(\tau))d\tau \int_a^b u(t,\tau)K_2(\tau, x(\tau))d\tau \qquad (4.2)$$

is well defined, and the existence of a solution of (4.1) is established upon the existence of a fixed point of T as defined in (4.2). The result below, which establishes the condition upon which the integral equation (4.1) has a unique solution, uses this fact.

Theorem 4.1: Assume that the following conditions are satisfied:

(i) $\gamma \le \frac{1}{2}$,

(ii) $\int_a^b u(t,\tau) \le 1$,

(iii) $\left|K_i(\tau, X(\tau)) - K_i(\tau, Y(\tau))\right| \le \frac{\sqrt{17}}{17}|x-y|, i = 1,2$ and

(iv) $\int_a^b u(t,\tau)\left|K_1(\tau, y(\tau) + K_2(\tau, x(\tau))\right|d\tau \le \sup_{t\in[a,b]} \frac{|x(t)-Tx(t)|}{|x(t) - y(t)|}$.

Then the integral equation (4.1) has a unique solution in X.

Proof:

$$G(Tx,Ty,Ty) = \left(2\sup_{t\in[a,b]}|Tx(t)-Ty(t)|\right)^2$$

$$= \gamma^2 2\sup_{t\in[a,b]}\Big(\Big|\int_a^b u(t,\tau)K_1(\tau, x(\tau))d\tau\int_a^b u(t,\tau)K_2(\tau, x(\tau))d\tau$$

$$-\int_a^b u(t,\tau)K_1(\tau, y(\tau))d\tau\int_a^b u(t,\tau)K_2(\tau, y(\tau))d\tau\Big|\Big)^2$$

$$\le \gamma^2 (2 \sup_{t\in[a,b]} \int_a^b u(t,\tau)|K_1(\tau,x(\tau)) - K_1(\tau,y(\tau))| d\tau \int_a^b u(t,\tau) K_2(\tau,x(\tau)) d\tau$$

$$+ \int_a^b u(t,\tau) K_1(\tau,x(\tau)) d_\tau \int_a^b u(t,\tau) |K_2(\tau,x(\tau)) - K_2(\tau,y(\tau))| d\tau |)^2$$

$$\le 4\gamma^2 (\sup_{t\in[a,b]} \sup_{t\in[a,b]} |K_1(\tau,x(\tau)) - K_1(\tau,y(\tau))| \sup_{t\in[a,b]} \int_a^b u(t,\tau) K_2(\tau,x(\tau)) d\tau |$$

$$+ \sup_{t\in[a,b]} \sup_{\tau\in[a,b]} |K_2(\tau,x(\tau)) - K_2(\tau,y(\tau))| \sup_{t\in[a,b]} \left| \int_a^b u(t,\tau) K_1(\tau,y(\tau)) d\tau \right|)^2$$

$$\le 4\gamma^2 (\frac{\sqrt{17}}{17} \sup_{t\in[a,b]} |x(t) - y(t)| \sup_{t\in[a,b]} | \int_a^b u(t,\tau) d\tau \int_a^b u(t,\tau) K_2(\tau,x(\tau)) d\tau$$

$$+ \int_a^b u(t,\tau) K_1(\tau,y(\tau)) d\tau \int_a^b u(t,\tau) d\tau |)^2$$

$$\le \frac{4}{17}\gamma^2 \sup_{t\in[a,b]} \left(\int_a^b u(t,\tau) d\tau \right)^2 (\sup_{t\in[a,b]} |x(t) - y(t)| \sup_{t\in[a,b]} | \int_a^b u(t,\tau) K_2(\tau,x(\tau)) d\tau$$

$$+ \int_a^b u(t,\tau) K_1(\tau,y(\tau)) d\tau |)^2$$

$$\le \frac{1}{17}\gamma^2 \left(2 \sup_{t\in[a,b]} |x(t) - y(t)| \sup_{t\in[a,b]} \int_a^b u(t,\tau) |K_1(\tau,y(\tau)) + K_2(\tau,x(\tau))| d\tau | \right)^2$$

$$\le \frac{1}{17} \left(2 \sup_{t\in[a,b]} |x(t) - y(t)| \sup_{t\in[a,b]} \frac{|x(t) - T(t)|}{|x(t) - y(t)|} \right)^2$$

$$= \frac{1}{17} \left(2 \sup_{t\in[a,b]} |x(t) - Tx(t)| \right)^2$$

$$= \frac{1}{17} G(x,Tx,Tx)$$

$$\le \frac{1}{17} [G(x,Tx,Tx) + G(y,Ty,Ty)]$$

Since the contraction map has a unique fixed point in Corollary 3.7, then theorem 4.1 has a unique solution.

Conclusion

This research provides conditions under which the Mann iteration sequence converges to the fixed points of Kannan, Chatterjea and Hardy and Rogers contractive mappings in convex G_b -metric spaces, where convexity is defined in the sense of Takahashi (1970). The fixed point theorems for these contractive mappings have been illustrated with examples. Additionally, the solution of an integral equation is obtained via the fixed point theorem for Kannan contraction mappings in convex G_b-metric spaces.

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