Approximation of the Jungck Multistep Iteration Process Generated by a Generalised Contractive-like Operator in Cone Banach Spaces

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Abstract
In this paper, we investigate an approximation theorem on the common fixed point of the Jungck multistep iteration process in a cone Banach space. The results obtained in this paper are improvements and generalisations of several results on common fixed points in the Literature.

Keywords: cone Banach space, contractive-like operators, convergence, weakly compatible maps

Introduction

The awareness of the concept of cone metric space is on the increase in the study of fixed point theory. Firstly, the concept was introduced in 2007 by Huang and Zhang as a generalisation of metric spaces, by replacing the real numbers with ordered Banach space. Secondly, several generalisations abound in the Literature. The results of Jungck *et al*. (2009) on common fixed points for weakly compatible pairs on cone metric spaces are generalisations as well as extensions of the result of Vetro (2007) where the normality assumption is removed. Arshad *et al*. (2009) proved a result on common fixed points for three self-mappings satisfying generalised contractive condition. Hence, results in non-normal cones are more general than normal ones.

Methods

Definitions and Lemmas

Definition 1: Sabetghadam, Masiha & Sanatpour (2009).
A cone P is a subset of a real Banach space E such that

(i) P is closed, nonempty and $P \neq \{0\}$;
(ii) If a, b are nonnegative real numbers and $x, y \epsilon P$, then $ax + by \epsilon P$;
(iii) $P \cap (-P) = \{0\}$.

For a given cone $P \subseteq E$, the partial ordering $\leq$ with respect to P is defined by $x \leq y$ if and only if $y - x \epsilon P$. The notation $x \ll y$ will stand for $y - x \epsilon\, int\, P$ where $int\, P$ denotes the interior of P. Also, we will use $x < y$ to indicate that $x \leq y$ and $x \neq y$.

The cone P is called normal if there exist a constant $M > 0$ such that for every $x, y \epsilon E$, if $0 \leq x \leq y$ then $||x|| \leq M||y||$. The least positive number satisfying this inequality is called the normal constant of P. The cone P is called regular if every increasing (or decreasing) and bounded above (or below) sequence is convergent in E.

Definition 2: Sabetghadam, Masiha & Sanatpour (2009).
Let X be a nonempty set and let E be a real Banach space equipped with the partial ordering $\leq$ with respect to the cone $P \subseteq E$. Suppose that the mapping $d: X \times X \to E$ satisfies the following conditions:

(d_1) $0 \leq d(x, y)$ for all $x, y \epsilon X$,
(d_2) $d(x, y) = 0$ if and only if $x = y$,
(d_3) $d(x, y) = d(y, x)$ for all $x, y \epsilon X$,
(d_4) $d(x, y) \leq d(x, z) + d(z, y)$ fr all $x, y, z \epsilon X$.

Then d is called a cone metric on X, and $(X, d)$is called a cone metric space. It is quite natural to consider Cone Normed Spaces (CNSs).

Definition 3: Karapinar & Turkoglu (2010).
Let X be a vector space over R. Suppose that the mapping $||.||_P : X \to E$ satisfies the following:

(N_1) $||x||_P \geq 0$ for all $x \epsilon X$,
(N_2) $||x||_P = 0$ if and only if $x = 0$,
(N_3) $||x + y||_P \leq ||x||_P + ||y||_P$ for all $x, y \epsilon X$,
(N_4) $||kx||_P = |k|||x||_P$ for all $k \epsilon R$.

Then $||.||_P$ is called a cone norm on X and $(X, ||.||_P)$ is called a cone normed space (CNS).

Definition 4: Sabetghadam, Masiha & Sanatpour (2009).
Let $X, ||.||_p)$ be a cone normed space, $x \epsilon X$ and $\{x_n\}_{n\geq 1}$ be a sequence in X. Then

(i) $\{x_n\}_{n\geq 1}$ converges to x, denoted by $\lim_{n\to\infty} x_n = x$, if for every $c \epsilon E$ with $0 \ll c$ there exist a natural number N such that $||x_n - x_m||_P \ll c$ for all $n \geq N$;
(ii) $\{x_n\}_{n\geq 1}$ is a Cauchy sequence if for every $c \epsilon E$ with $0 \ll c$ there exists a natural number N such

that $||x_n - x_m||_P \ll c$ for all $n, m \geq N$.

A complete cone-normed space is called a cone Banach space.

Definition 5: Abass & Jungck (2008).
A point $x \epsilon X$ is called a coincidence point of self maps S, T if there exists a point q (called a point of coincidence) in X such that $p = Sq = Tq$. Self-maps S and T are said to be weakly compatible if they commute at their coincidence point, that is, if $Sx = Tx$ for some $x \epsilon X$, then $STx = TSx$.

We now state some of the Jungck-type iteration process known in Literature in a cone Banach space. Let $(X, ||.||_P)$ be a cone Banach space and $S, T : X \times X \to E$ be two mappings such that $T(X) \subseteq S(X)$.
For any $x_0 \epsilon X$, the Jungck iterative scheme is defined as the sequence $\{Sx_n\}_{n=1}^{\infty}$ such that

$$Sx_{n+1} = Tx_n, \quad n \geq 0. \tag{1}$$

Singh *et al*. (2005) recently introduced the Jungck–Mann iterative process and discussed its stability for a pair of contraceptive maps. The iterative process is defined in a cone Banach space as follows:
For any given $u_0 \epsilon X$, the Jungck–Mann iterative scheme $\{Sx_n\}_{n=1}^{\infty}$ is defined by

$$Sx_{n+1} = (1 - \alpha_n)Sx_n + \alpha_n Tx_n \tag{2}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ is a real sequence in [0,1] such that $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Olatinwo and Imoru (2008) and Olatinwo (2008) built on the work of Singh *et al.* (2005) to introduce the Jungck–Ishikawa and Jungck–Noor iterative schemes and used their convergences to approximate the coincidence points of some pairs of generalised contractive-like operators in Banach spaces with the assumption that each one of the pairs of maps is injective. The iterative schemes are defined on a cone Banach space as follows:

For $x_0 \epsilon X$, the Jungck–Ishikawa iterative scheme is the sequence $\{Sx_n\}_{n=0}^{\infty}$ defined by

$$\begin{aligned} Sx_{n+1} &= (1 - \alpha_n)Sx_n + \alpha_n Ty_n, \\ Sy_n &= (1 - \beta_n)Sx_n + \beta_n Tx_n \end{aligned} \tag{3}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are real sequences in [0,1) such that $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Let $x_0 \epsilon X$, the Jungck–Noor iterative scheme is the sequence $\{Sx_n\}_{n=0}^{\infty}$ defined by

$$\begin{aligned} Sx_{n+1} &= (1 - \alpha_n)Sx_n + \alpha_n Ty_n, \\ Sy_n &= (1 - \beta_n)Sx_n + \beta_n Tz_n, \\ Sz_n &= (1 - \gamma_n)Sx_n + \gamma_n Tx_n \end{aligned} \tag{4}$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are real sequences in [0,1] such that $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Olaleru and Akewe (2010) introduced the Jungck multistep iteration process and used the scheme to approximate the common fixed point of a pair of weakly compatible maps in a Banach space. The iterative scheme is defined in a cone Banach space as follows:
Let $x_0 \epsilon X$, the Jungck multistep iterative process is the sequence $\{Sx_n\}_{n=0}^{\infty}$ defined by

$$\begin{aligned} Sx_{n+1} &= (1 - \alpha_n)Sx_n + \alpha_n Ty_n^1 \\ Sy_n^i &= (1 - \beta_n^i)Sx_n + \beta_n^i Ty_n^{i+1}, \; i = 1,2,\dots,k-2 \\ Sy_n^{k-1} &= (1 - \beta_n^{k-1})Sx_n + \beta_n^{k-1} Tx_n, \; k \geq 2 \end{aligned} \tag{5}$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n^i\}_{n=0}^{\infty}$, $i = 1,2,\dots,k-1$ are real sequences in [0,1) such that $\sum_{n=0}^{\infty} \alpha_n = \infty$.

The Jungck multistep iterative process defined in (5) is remarkable because it is a generalisation of the Jungck–Noor (4). Jungck–Ishikawa (3), Jungck–Mann (2) and Jungck (1) iteration processes, provided the maps S and T are weakly compatible.

We now attempt to give the cone version of some contractive operators necessary in this work.
Let $(X, ||.||_P)$ be a cone Banach space, there exist $S, T: X \times X \to E$ such that $T(X) \subseteq S(X)$ for $x, y \epsilon X$ and $h \epsilon (0,1)$:

$$\left||Tx - Ty|\right|_P \leq h \max\left\{\left||Sx - Sy|\right|_P, \frac{||Sx-Tx||_P + ||Sy-Ty||_P}{2}, \frac{||Sx-Ty||_P + ||Sy-Ty||_P}{2}\right\} \tag{6}$$

$$||Tx - Ty||_P \leq h \max\left\{\left||Sx - Sy|\right|_P, \frac{||Sx-Tx||_P + ||Sy-Ty||_P}{2}, Sx - Ty| \quad |_P, \left||Sy - Tx|\right|_P\right\} \tag{7}$$

$$||Tx - Ty||_P \leq \delta\left||Sx - Sy|\right|_p + L\left||Sx - Tx|\right|_p, \; L > 0, \; 0 < \delta < 1 \tag{8}$$

$$||Tx - Ty||_P \leq \frac{\delta||Sx-Sy||_P + \varphi(||Sx-Tx||_P)}{1 + M||Sx-Tx||_P}, \; 0 \leq \delta < 1, \; M \geq 0 \tag{9}$$

$$||Tx - Ty||_P \leq \delta\left||Sx - Sy|\right|_p + \varphi\left(\left||Sx - Tx|\right|_p\right), \; 0 \leq \delta < 1 \tag{10}$$

Comparing (6)–(10), we have the following:

Proposition: (6) $\Rightarrow$ (7) $\Rightarrow$(8) $\Rightarrow$ (9) $\Rightarrow$ (10) but the converses are not true.

Proof: (6) $\Rightarrow$(7): This is trivial since $\frac{||Sx-Tx||_P + ||Sy-Ty||_P}{2} \leq \max\{\left||Sx - Ty|\right|_P + \left||Sy - Tx|\right|_P\}$

(7) $\Longrightarrow$ (8): we consider each possibility.

Case 1: Suppose

$$||Tx - Ty||_P \leq h||Sx - Ty||_P$$
$$\leq h||Sx - Tx||_P + ||Tx - Ty||_P$$

hence $||Tx - Ty||_P \leq \frac{h}{1-h}||Sx - Tx||_P$

where $L = \frac{h}{1-h}$. This ends the proof.

Case 2: Suppose

$$||Tx - Ty||_P \leq h\frac{||Sx - Tx||_P + ||Sy - Ty||_P}{2}$$
$$\leq h\frac{||Sx - Tx||_P + ||Sy - Sx + Sx - Tx + Tx - Ty||_P}{2}$$
$$\leq h||Sx - Tx||_P + \frac{h}{2}||Sy - Sx||_P + \frac{h}{2}||Tx - Ty||_P$$

Thus,

$$||Tx - Ty||_P \leq \frac{h}{2-h}||Sy - Sx||_P + \frac{2h}{2-h}||Sx - Tx||_P$$

where $\delta = \frac{h}{2-h}$ and $L = \frac{2h}{2-h}$. This ends the proof.

Case 3: Suppose $||Tx - Ty||_P \leq h||Sy - Tx||_P \leq h||Sy - Sx||_P + h||Sx - Tx||$.

(8) $\Longrightarrow$ (9): Suppose $M = 0$ and $\varphi(t) = Lt$ in (9), we have (8).

(9) $\Longrightarrow$ (10): This immediately follows from the fact that

$$||Tx - Ty||_P \leq \frac{\delta||Sx - Sy||_P + \varphi(||Sx - Tx||_P)}{1 + M||Sx - Tx||_P}$$
$$\leq \delta||Sx - Sy||_P + \varphi(||Sx - Tx||_P$$

This ends the proof.

Lemma: Olaleru and Akewe (2010).

Let $\{\theta_n\}_{n=0}^{\infty}$ be a sequence of nonnegative numbers satisfying $\theta_{n+1} \leq \left(1 - \lambda_n\right)\theta_n,\ n \geq 0,$

Where $\lambda_n \epsilon\ [0,1)\ and\ \sum_{n=0}^{\infty}\lambda_n = \infty$.

$Then\ \lim_{n\to\infty}\theta_n = 0.$

Results and Discussion

Theorem 1: Let $(X, ||.||_P)$ be a cone Banach space and $S, T: X \times X \to E$ such that

$$||Tx - Ty||_P \leq \delta||Sx - Sy||_P + \varphi(||Sx - Tx||_P)$$

where $0 \leq \delta < 1$ and $\varphi(t)$ a monotonic increasing and continuous function, and $T(X) \subseteq S(X)$. Assume S and T have a coincidence point z such that $Tz = Sz = p$. For any $x_0 \epsilon Y$, the Jungck multistep iteration (5) $\{Sx_n\}_{n=1}^{\infty}$ converges to p. Further, if S, T commute at p (i.e., S and T are weakly compatible) then p is the unique common fixed point of S, T.

Proof. In view of (10) and (5) coupled with the fact that $Tz = Sz = p$, we have

$$||Sx_{n+1} - p||_P \leq (1 - \alpha_n)||Sx_n - p||_P + \alpha_n||Tz - Ty_n^1||_P$$
$$\leq (1 - \alpha_n)||Sx_n - p||_P + \alpha_n[\delta||Sz - Sy_n^1||_P + \varphi\left(||Sz - Tz||_P\right)]$$
$$= (1 - \alpha_n)||Sx_n - p||_P + \delta\alpha_n||p - Sy_n^1||_P \tag{11}$$

An application of (10) and (5) gives $||Sy_n^1 - p||_P \leq (1 - \beta_n^1)||Sx_n - p||_P + \beta_n^1||Tz - Ty_n^2||_P$

$$\leq (1 - \beta_n^1)||Sx_n - p||_P + \beta_n^1\left[\delta||Sz - Sy_n^2||_P + \varphi\left(||Sz - Tz||_P\right)\right] \tag{12}$$

Substituting (12) in (11), we have

$$||Sx_{n+1} - p||_P \leq (1 - \alpha_n)||Sx_n - p||_P + \delta\alpha_n(1 - \beta_n^1)||Sx_n - p||_P + \delta^2\alpha_n\beta_n^1||Sy_n^1 - p||_P$$
$$= (1 - (1 - \delta)\alpha_n - \delta\alpha_n\beta_n^1)||Sx_n - p||_P + \delta^2\alpha_n\beta_n^1||Sy_n^1 - p||_P \tag{13}$$

Similarly, an application of (10) and (5) gives

$$||Sy_n^2 - p||_P \leq (1 - \beta_n^2)||Sx_n - p||_P + \beta_n^2||Tz - Ty_n^3||_P$$
$$\leq (1 - \beta_n^2)||Sx_n - p||_P + \beta_n^2[\delta||Sz - Sy_n^3||_P + \varphi\left(||Sz - Tz||_P\right)] \tag{14}$$

Substituting (14) in (13), we have

$$||Sx_{n+1} - p||_P \leq (1 - (1 - \delta)\alpha_n - \delta\alpha_n\beta_n^1)||Sx_n - p||_P$$
$$+\delta^2\alpha_n\beta_n^1(1 - \beta_n^2)||Sx_n - p||_P + \delta^3\alpha_n\beta_n^1\beta_n^2||Sy_n^3 - p||_P$$
$$= (1 - (1 - \delta)\alpha_n - (1 - \delta)\delta\alpha_n\beta_n^1 - \delta^2\alpha_n\beta_n^1\beta_n^2)||Sx_n - p||_P + \delta^3\alpha_n\beta_n^1\beta_n^2||Sy_n^3 - p||_P \tag{15}$$

Similarly, an application of (10) and (5) gives

$$||Sy_n^3 - p||_P \leq (1 - \beta_n^3)||Sx_n - p||_P + \delta\beta_n^3||Sy_n^4 - p||_P \tag{16}$$

Substituting (16) in (15), we have

$$
\begin{aligned}
||Sx_{n+1}-p||_P &\le (1-(1-\delta)\alpha_n-(1-\delta)\delta\alpha_n\beta_n^1-\delta^2\alpha_n\beta_n^1\beta_n^2)||Sx_n-p||_P \\
&\quad+\delta^3\alpha_n\beta_n^1\beta_n^2(1-\beta_n^3)\big||Sx_n-p|\big|_P+\delta^4\alpha_n\beta_n^1\beta_n^2\beta_n^3||Sy_n^4-p||_P \\
&=(1-(1-\delta)\alpha_n-(1-\delta)\delta\alpha_n\beta_n^1-(1-\delta)\delta^2\alpha_n\beta_n^1\beta_n^2 \\
&\quad-\delta^3\alpha_n\beta_n^1\beta_n^2\beta_n^3))\big||Sx_n-p|\big|_P+\delta^4\alpha_n\beta_n^1\beta_n^2\beta_n^3)||Sy_n^4-p||_P \\
&\le (1-(1-\delta)\alpha_n-\delta^3\alpha_n\beta_n^1\beta_n^2\beta_n^3))\big||Sx_n-p|\big|_P+\ \delta^4\alpha_n\beta_n^1\beta_n^2\beta_n^3)||Sy_n^4-p||_P \qquad (17)
\end{aligned}
$$

Continuing the above process we have

$$
\begin{aligned}
||Sx_{n+1}-p||_P &\le (1-(1-\delta)\alpha_n-\delta^{k-2}\alpha_n\beta_n^1\beta_n^2\beta_n^3\dots\beta_n^{k-2}))||Sx_n-p||_P \\
&\quad+\delta^{k-1}\alpha_n\beta_n^1\beta_n^2\beta_n^3\dots\beta_n^{k-2})||Sy_n^{k-1}-p||_P \\
&\le (1-(1-\delta)\alpha_n-\delta^{k-2}\alpha_n\beta_n^1\beta_n^2\beta_n^3\dots\beta_n^{k-2}))||Sx_n-p||_P \\
&\quad+\delta^{k-1}\alpha_n\beta_n^1\beta_n^2\beta_n^3\dots\beta_n^{k-2})[(1-\beta_n^{k-1})\big||Sx_n-p|\big|_P+\beta_n^{k-1}\big||Tz-Tx_n|\big|_P] \\
&\le (1-(1-\delta)\alpha_n-\delta^{k-2}\alpha_n\beta_n^1\beta_n^2\beta_n^3\dots\beta_n^{k-2}))||Sx_n-p||_P \\
&\quad+\delta^{k-1}\alpha_n\beta_n^1\beta_n^2\beta_n^3\dots\beta_n^{k-2})[(1-\beta_n^{k-1})\big||Sx_n-p|\big|+\delta\beta_n^{k-1}\big||Sx_n-p|\big|_P] \\
&\le (1-(1-\delta)\alpha_n-\delta^{k-2}\alpha_n\beta_n^1\beta_n^2\beta_n^3\dots\beta_n^{k-2})\ +\ \delta^{k-1}\alpha_n\beta_n^1\beta_n^2\beta_n^3\dots\beta_n^{k-2})||Sx_n-p||_P \\
&\le (1-(1-\delta)\alpha_n)\big||Sx_n-p|\big|_P\ \le \big(1-(1-\delta)\big)\big||Sx_n-p|\big|_P
\end{aligned}
$$

Hence $Sx_n \to p$ since $1-\delta<1$ for all n.

Next, we show that p is unique.
Suppose there exists another point of coincidence p^*. Then there is an $z^* \epsilon\, X$ such that $Tz^*=Sz^*=p^*$. Hence, we have

$$||z-z^*||_P=||Tz-Tz^*||_P\le\delta\big||Sz-Sz^*|\big|_P+\varphi\left(\big||Sz-Tz|\big|_P\right)=\delta||z-z^*||_P$$

Since S,T are weakly compatible, then $TSz=STz$ and so $Tp=Sp$. Hence p is a coincidence point of S,T and hence the coincidence point is unique, then $p=z$ and hence $Sp=Tp=p$ and therefore p is the unique common fixed point of S,T and the proof is complete.
Theorem 1 leads to the following corollaries:

Corollary 1: Theorem 2 of Olaleru and Akewe (2010).
Let $(X,||.||)$ be a Banach space and $S,T\colon Y\to X$ such that $\big||Tx-Ty|\big|\le\delta\big||Sx-Sy|\big|+\varphi(\big||Sx-Tx|\big|)$ holds where $0\le\delta<1$ and $\varphi(t)$ a monotonic increasing and continuous function and $T(Y)\subseteq S(Y)$. Assume S and T have a coincidence point z such that $Tz=Sz=p$. For any $x_0\,\epsilon\,Y$, the Jungck multistep iteration (5) $\{Sx_n\}_{n=1}^{\infty}$ converges to p. Further, if $Y=XS,T$ commute at p (i.e., S and T are weakly compatible) then p is the unique common fixed point of S,T.

Remark 1: Weaker versions of Corollary 1 are a result of Olatinwo (2008) where S is assumed injective and the convergence is not to the common fixed point but to the coincidence point of S,T. Furthermore, the Jungck multistep iteration used in Theorem 2 is more general than the Jungck–Ishikawa and Jungck–Noor iteration used in Olatinwo (2008). It is already shown in Olatinwo and Imoru (2008), that if $S(Y)$ or $T(Y)$ is a complete subspace of X, then the maps satisfying the generalised Zamfirescu condition

$$||Tx-Ty||_P\ \le\delta\big||Sx-Sy|\big|_P+2\delta||Sx-Tx||_P$$

has a unique coincidence point. Hence we have the following results.

Theorem 3: Let $(X,||.||)$ be a cone Banach space and $S,T\colon X\times X\to E$ such that

$$||Tx-Ty||_P\le\delta\big||Sx-Sy|\big|_P+2\delta||Sx-Tx||_P$$

where $0\le\delta<1$ is satisfied and $T(X)\subseteq S(X)$. Assume S and T are weakly compatible. For any $x_0\,\epsilon\,X$, the Jungck-multistep iteration process (5) $\{Sx_n\}_{n=1}^{\infty}$ converges to the unique common fixed point of S,T.

Since the Jungck–Noor, Jungck–Ishikawa and Jungck–Mann iterations are special cases of the Jungck multistep iteration process then we have the following consequences.

Remark 2: Let $(X,||.||)$ be a cone Banach space and $S,T\colon X\times X\to E$ such that the generalised Zamfirescu condition

$$\big||Tx-Ty|\big|_P\le\delta\big||Sx-Sy|\big|_P+2\delta\big||Sx-Tx|\big|_P$$

where $0\le\delta<1$ is satisfied and $T(X)\subseteq S(X)$. Assume S and T are weakly compatible. For any $x_0\,\epsilon\,E$, the Jungck–Noor iterative scheme (3) $\{Sx_n\}_{n=1}^{\infty}$ converges to the unique common fixed point of S,T.

Corollary 3: Let $(X,||.||)$ be a cone Banach space and $S,T\colon X\times X\to E$ such that

$$||Tx-Ty||_P\le\delta\big||Sx-Sy|\big|_P+2\delta||Sx-Tx||_P$$

where $0\le\delta<1$ is satisfied and $T(X)\subseteq S(X)$. Assume S and T are weakly compatible. For any

$x_0 \epsilon E$, the Jungck–Ishikawa iterative scheme (3) $\{Sx_n\}_{n=1}^{\infty}$ converges to the unique common fixed point of S, T.

Corollary 4: Theorem 2 of Arshad, Azam and Vectro (2009).
Let E be a Banach space and $S, T: X \times X \to E$ such that $||Tx - Ty||_P \leq \delta||x - y||_p + 2\delta||x - Tx||_P$ where $0 \leq \delta < 1$ is satisfied. For any $x_0 \epsilon X$, the Ishikawa iterative scheme (5) $\{x_n\}_{n=1}^{\infty}$ converges to the unique fixed point of T.

Remark 2:

(i) A weaker version of Corollary 3 is the main result of Olatinwo and Imoru (2008), where the convergence is to the coincidence point of S, T and S is assumed injective.

(ii) If $S = I_d$ in Corollary 4, then we have the main result of Berinde (2004).

Corollary 5: Let E be a Banach space and $S, T : E \to E$ such that

$$||Tx - Ty||_P \leq \delta||Sx - Sy||_P + 2\delta Sx - Tx||_P$$

where $0 \leq \delta < 1$ is satisfied and $T(E) \subseteq S(E)$. Assume S and T are weakly compatible. For any $x_0 \epsilon E$, the Jungck–Mann iteration (2) $\{Sx_n\}_{n=1}^{\infty}$ converges to the unique common fixed point of S, T.

Corollary 6: Let E be a Banach space and $S, T : E \to E$ such that the contractive condition

$$||Tx - Ty||_P \leq \delta||x - Sy||_P + L||x - Tx||_P$$

where $0 \leq \delta < 1$ and $L \geq 0$ is satisfied. For any $x_0 \epsilon E$, the Mann iteration (Mann 1953), $\{x_n\}_{n=1}^{\infty}$ converges to the unique fixed point of T.

Remark 3: If $S = I_d$, the corollary gives the result of Olaleru (2006). It is already shown in Abbas and Jungck (2008) and Olaleru (2009), that if $S(Y)$ or $T(Y)$ is a complete subspace of X, then maps S and T satisfying

$$||Tx - Ty||_P \leq h\, max\left\{||Sx - Sy||_P, \frac{||Sx - Tx||_P + ||Sy - Ty||_P}{2}, ||Sx - Ty||_P, ||Sy - Tx||_P\right\}$$

where $0 \leq \delta < 1$, have a unique coincidence point. Hence we have the following results.

Theorem 4: Let E be a Banach space and $S, T : E \to E$ such that

$$||Tx - Ty||_P \leq h\, max\left\{||Sx - Sy||_P, \frac{||Sx - Tx||_P + ||Sy - Ty||_P}{2}, ||Sx - Ty||_P, ||Sy - Tx||_P\right\}$$

Assume S and T are weakly compatible. For any $x_0 \epsilon E$, the Jungck multistep iteration (5) $\{Sx_n\}_{n=1}^{\infty}$ converges to the unique common fixed point of S, T.
Since the Jungck–Noor, Jungck–Ishikawa and Jungck–Mann iterations are special cases of hybrid-multistep iteration then we have the following consequences.

Corollary 7: Let E be a Banach space and $S, T : E \to E$ such that

$$||Tx - Ty||_P \leq h\, max\left\{||Sx - Sy||_P, \frac{||Sx - Tx||_P + ||Sy - Ty||_P}{2}, ||Sx - Ty||_P, ||Sy - Tx||_P\right\}$$

and $T(E) \subseteq S(E)$. Assume S and T are weakly compatible. For any $x_0 \epsilon E$, the Jungck–Noor iteration (4) $\{Sx_n\}_{n=1}^{\infty}$ converges to the unique common fixed point of S, T.

Corollary 8: Let E be a Banach space and $S, T : E \to E$ such that

$$||Tx - Ty||_P \leq h \max\left\{||Sx - Sy||_P, \frac{||Sx - Tx||_P + ||Sy - Ty||_P}{2}, ||Sx - Ty||_P, ||Sy - Tx||_P\right\}$$

and $T(E) \subseteq S(E)$. Assume S and T are weakly compatible. For any $x_0 \epsilon E$, the Jungck–Ishikawa iteration (3) $\{Sx_n\}_{n=1}^{\infty}$ converges to the unique common fixed point of S, T.

Corollary 9: Let E be a Banach space and $S, T : E \to E$ such that

$$||Tx - Ty||_P \leq h \max\left\{||Sx - Sy||_P, \frac{||Sx - Tx||_P + ||Sy - Ty||_P}{2}, ||Sx - Ty||_P, ||Sy - Tx||_P\right\}$$

and $T(E) \subseteq S(E)$. Assume S and T are weakly compatible. For any $x_0 \epsilon E$, the Jungck–Mann iteration (2) $\{Sx_n\}_{n=1}^{\infty}$converges to the unique common fixed point of S, T.

Example 1: Let $Y = ([0.2], |.|)$. Define T and S by

$$Tx = \begin{cases} \frac{1}{2}, & \text{if} \quad x \epsilon (0,1] \\ 0, & \text{if} \quad x \epsilon \{0\} \cup (1,2] \end{cases} \quad \text{and} \quad Sx = \begin{cases} 0, & \text{if} \quad x = 0 \\ x+1, & \text{if} \quad x \epsilon (0,1] \\ x-1, & \text{if} \quad x \epsilon (1,2] \end{cases}$$

$$||Tx - Ty|| \leq \delta||Sx - Sy|| + \varphi(||Sx - Tx||),$$

where $\delta = \frac{1}{2}$ and $\varphi(t) = 2\delta t$. $T(Y) = \{0\} \cup \{\frac{1}{2}\}$ and $S(Y) = [0,2]$. Then $T(Y) \subseteq S(Y)$. It is easy to easy that $S(0) = T(0) = 0$ and $ST(0) = S(0) = 0$, $TS(0) = T(0) = 0$. Hence, the common fixed point of $S$and T is 0.

Running a MATLAB 7.10.0 script, with $a_n = \frac{2}{3}, b_n = c_n = \frac{1}{2n+4}, a_n^1 = b_n^1 = c_n^1 = \frac{1}{4}$ for all $n > 0$ and $x_0 = 1$ we have the following results:

$$\begin{aligned} Sx_1 &= 0.750000000000000 \\ Sx_2 &= 0.187500000000000 \\ Sx_3 &= 0.046875000000000 \\ Sx_4 &= 0.011718750000000 \\ Sx_5 &= 0.002929687500000 \\ Sx_6 &= 0.000732421875000 \\ Sx_7 &= 0.000183105468750 \\ Sx_8 &= 0.000045776367188 \\ Sx_9 &= 0.000011444091797 \\ Sx_{10} &= 0.0000002861022949 \end{aligned}$$

We notice that $\{Sx_n\}$ in (5) converges to 0, which is the common fixed point of S and T.

Example 2: Let $(X, d) = ([0,10], |.|)$. Define S and T by

$$Sx = \begin{cases} 3, & \text{if} \quad x \epsilon (0,2] \\ 0, & \text{if} \quad x \epsilon \{0\} \cup (2,10] \end{cases} \quad \text{and} \quad Tx = \begin{cases} 0, & \text{if} \quad x = 0 \\ x+8, & \text{if} \quad x \epsilon (0,2] \\ x-2, & \text{if} \quad x \epsilon (2,10] \end{cases}$$

Then $Sx = Tx$ iff $x = 0$ and $ST(0) = T(0) = 0$, $TS(0) = S(0) = 0$.
Therefore S and T are weakly compatible.

Conclusion

This paper used the Jungck multistep iteration process to approximate the common fixed point of a pair of generalised contractive like operators in a cone Banach space and justified the claim with examples. Consequently several results in the Literature are generalised and extended.

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