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Combinatorial Interactions: Partition of Integers, Permutation Groups, and Nilpotent Orbits of Type A

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Abstract: Partition of integers has various and extensive applications in divers area of Mathematics such as Combinatorics, Representation theories and Algebraic geometry. In this paper we gives an expository remark on some notable combinatorial interactions between partition of integers, group of permutations S_n and nilpotent orbits of type A. (where the underlying group defines the type and here the underlying group is the general linear group). Some of the results include counting of nilpotent orbits in type A, the cycle structure of elements of group of permutations S_n and enumeration of irreducible $S_n -$ module.

1. Introduction

The genesis of theories of integer partitions is incomplete without mentioning some past heroes such as Euler, Ramanujan, Legendre, Hardy, Selberg and so on. Their immerse contributions in this area of Mathematics led to some topics such as Partition identities and bijections, Young diagram, partition generating functions, q-binomial number, partition congruences (Ian Grant Macdonald, 1998). In this article, we survey some connections between partition of integers, group of permutations S_n and nilpotent orbits of type A. The rest of this section will be devoted to definition of some basic terms as relevant to our discussion in the sequel. In section 2, we discuss some connections between integer partitions and the group of permutations. In section 3, we look at the connection between partition of integers and the nilpotent orbit in type A.

For the sake of completeness, we shall first of all discuss partition of integer $n > 0$ and dominance order on partitions.

Definition 1.1. A partition λ of non negative integer n written as $\lambda \vdash n$, is a sequence $\lambda = (\lambda_i)_{i=1}^{k}$ of integers such that $\lambda_1, \lambda_2 \ldots, \lambda_k$ and $\sum_{i=1}^{k} \lambda_i = n$. Each λ_i is called part of λ. The number of parts is called the length of λ denote by $l\lambda$, and the sum of parts is the weight of λ denoted by $|\lambda| = \lambda_1, \lambda_2 \ldots, \lambda_k$.

Let $n = 8$, then $\lambda = (3,2,2,1)$ is one of the partitions of 8, $l(\lambda)$=4 and $|\lambda|$=8.

Example: We denote the set of all partitions of n by P(n) and the set of partitions by P. To avoid repetition of parts in λ , we use indices to record multiplicity in partitions λ and write $\lambda = \lambda_1^{a_1}, \lambda_2^{a_2}, \ldots, \lambda_k^{q_k}$, if λ_i $(1 \le i \le k)$ appears in a_i times in λ and we refer to a_i as the multiplicity of λ_i .

For example,

$$\lambda = (2,2) = (2^2)$$

Thus

$$P(5) = \{(5), (4,1), (3,2), (3, 1^2), (2^2, 1), (2, 1^3), (1^5)\} \quad (1.1)$$

In order to avoid double counting, we take (4,1) and (1,4) as the same partition of 5. Below is a table of $n \leq 14$ and its corresponding |P(n)|. We take $P(n) = 0$ for all $n < 0$ and $P_0 = 1$.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14
|P(n)|	1	2	3	4	7	11	15	22	30	42	56	77	101	135

TABLE 1: Table of $P(n), 1 \leq n \leq 14$

Remark 1.1

i. If a partition ⋏ is of the form $⋏ = (n-1, n-2, ..., 2,1)$ then it is called staircase partition, which we shall denote by $⋏_n$.
ii. A partition of the form $⋏ = (m, m, m, ..., m)$, where $m \epsilon N$ is called block partition. For instance, let n=20 and m=4, then ⋏=(4,4,4,4).
iii. But if ⋏ is of the form $⋏ = n-k,\, 1^k, : k \geq 1$, then ⋏ is called hook partition. Let n=7 and k=4, then ⋏=(3,1,1,1,1).

Definition 1.2 The generating function in x denoted f(x) for a sequence $(a_n)_{n\leq 0}$ is the power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \quad (1.2)$$

.

The generating function for P(n) is given as

$$f(x) = \sum_{n=0}^{\infty} \#P(n)x^n = \prod_{n=1}^{\infty} \frac{1}{1-x^n} \,. \quad (1.3)$$

Denote by $P_d(n)$ the set of partitions with distinct parts and $P_0(n)$ the set of partitions with odd parts. For $n = 5$, with

$$P(5) = \{(5),(4,1),(3,2),(3,1^2),(2^2,1),(2,1^3)(1^5)\} \quad (1.4)$$

we have

$$P_0(5) = \{(5),(3,1,1),(1,1,1,1,1)\} \quad (1.5)$$

and

$$P_0(5) = \{(5),(4,1),(3,2)\} \quad (1.6)$$

with

$$\#P_0(5) = 3 = \#P_d(5). \quad (1.7)$$

Remark 1.2 For any $n > 0$: $P_0(n) = P_d(n)$. Denote by $P_k(n)$ the set of partitions with exactly k ($\geq$ 1) parts, y taking conjugates of each $⋏ \in P(n)$ one would see that $\#P_k(n)$ is equal to the number of partitions in which the largest part is k. The values of $\#P_k(n)$ satisfies the recurrence relation

$$\#P_k(n) = \#P_k(n-k) + \#P_{k-1}(n-1), k \in N \,. \quad (1.8)$$

With the initial value $\#P_0(0) = 1$ and $\#P_k(k) = 1$. The connection between $\#P(n)$ and $\#P_k(n)$ is revealed in the equation below

$$\#P(n) = \sum_{k=0}^{n} \#P_k(n) \ . \tag{1.9}$$

Fixing k and varying n, the generating functions for the number of partitions with exactly k parts is

$$\sum_{n\geq 0} P_k(n)x^n = x^k . \prod_{n=1}^{k} \frac{1}{1-x^k} \ . \tag{1.10}$$

Following Ian Grant Macdonald, (1998), we define ordering on $P(n)$ as follows:

Definition 1.3 Let $\lambda, \mu \in P$. We say λ contains μ and write $\mu \subset \lambda$, $if\ \mu_i \leq \lambda_i$ for all $i \in n$.

Example 1.2 Let $\lambda = (3,2,2,1) and\ \mu = (2,1,1,1)$. then $\mu \subset \lambda$.

Definition1.4 Let $L_n \in P(n) \times P(n)$ denote the reverse lexicographical ordering on the set $P(n)$ of partitions of n. We say $(\lambda, \mu) \in L_n$ if either $\lambda = \mu$ or the first non vanishing difference $\lambda_i - \mu_i > 0$.
L_n is a total ordering.

Example 1.3 When $n = 6$, L_6 arranges $P(6)$ in the sequence.

$$(6), (5,1), (4,2), (4,1,1), (3,3), (3,2,1), (3,1^3), (2^3), (2^2 1^2), (2,1^4), (1^6).$$

We define mother ordering $L_n^{'}$ on $P(n)$ as follows:

Definition 1.5 The ordering $L_n^{'}$ on $P(n)$ is the set of all (λ, μ) such that either $\lambda = \mu$ or the first non vanishing difference $\lambda_i^* - \mu_i^* < 0$, where $\lambda_i^* = \lambda_{n-i+1}$.

Example 1.4. Let $\lambda = (4, 1^4) and\ \mu = 2^4$ then $(\lambda, \mu) \in L_8^{'}$.

Theorem 1.1[4] Let $\lambda, \mu \in P(n)$. Then $(\lambda, \mu) \in L_n^{'} \leftrightarrow (\lambda^{'}, \mu^{'}) \in L_n$.

Definition 1.6. Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ and $\mu = (\mu_1, \mu_2, \ldots)$ be partitions of n. We say λ dominates μ and write $\lambda \trianglerighteq \mu$. If for any $k \geq 1$.

$$\sum_{i=1}^{k} \lambda_i \geq \sum_{i=k}^{n} \mu_i \ . \tag{1.11}$$

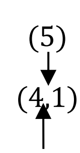
This ordering is called dorminant partial order on partitions of some fixed $n \in N$. For example, in $P(5)$, the relations below hold:

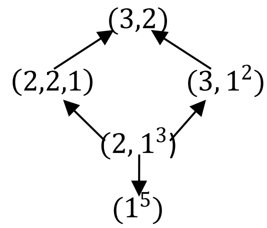
$$(5) \trianglerighteq (4,1) \trianglerighteq (3,2) \trianglerighteq (3,1,1) \trianglerighteq (2,2,1) \trianglerighteq (2,1,1,1) \trianglerighteq (1,1,1,1,1) \tag{1.12}$$

Theorem 1.1[Ian Grant Macdonald, 1998] Let $\lambda, \mu \in P(n)$. Then $\lambda \geq \mu\ if\ and\ only\ if\ \mu^{'} \geq \lambda^{'}$.

Remark 1.3

The orderings $L_n, L_n^{'}$ are distinct for $n \geq 6$. For example, if $\lambda = (3^3)$ and $\mu = (2^3)$ we have $(\lambda, \mu) \in L_6$ and $(\mu, \lambda) \in L_6^{'}$. Hence if $\lambda, \mu \in$ P(n), then $(\lambda, \mu) \in L_6 \leftrightarrow (\mu, \lambda) \in L_6^{'}$.
Note that, for any partial ordered set S, there exists a diagram, called Hasse diagram through which S could be visualized. Below is a Hasse diagram for P(5).





The Symmetric Group: In this subsection, we give definitions of some basic terms on the group of permutation which shall be needed in the sequel.

Definition 1.7: The collection of all permutations, denoted by S_X forms a group, called the group of permutations, under the operation 'composition' (Adetunji, 2023)

Remark 1.4 If X= [n] = {1,2,3, … n}, then we replace S_X by S_n.
There are different ways to represent elements of S_n. One of these ways is referred to as two line notations. For instance, consider a permutation $\sigma \in S_7$ with $\sigma(1) = 2, \sigma(2) = 5, \sigma(3) = 6, \sigma(4) = 4, \sigma(5) = 7, \sigma(6) = 3, \sigma(7) = 1$. Then σ is written as

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \sigma(1) & \sigma(2) & \sigma(3) & \sigma(4) & \sigma(5) & \sigma(6) & \sigma(7) \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 5 & 6 & 4 & 7 & 3 & 1 \end{pmatrix} \quad (1.13)$$

We multiply permutations from right to left. Thus $\pi\sigma$ is the bijection obtained by first applying σ, followed by π.

Example 1.5 . If $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 5 & 4 & 2 \end{pmatrix}$, $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 1 & 2 & 5 \end{pmatrix}$. Then $\sigma\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 5 & 2 & 3 \end{pmatrix}$

If we write only entries in the second row of the array, then what we have is one line notation which is another interesting way to represent elements of S_n. For example, the one line notation of σ above is 43125

.

Remark 1.5 The symmetric group of degree n is the symmetric group on the set [n] = { 1, 2, ..., n }. S_n has order n!, it is Abelian if and only if $n \leq 2$. We shall denote the identity element of S_n by e. Another way to represent elements of S_n is to write them in cycle form. In fact some of the interesting applications of group of permutations are revealed when they are expressed in cycle form. We shall hence, due to this, take little of our time to discuss cycle structure of elements of S_n in what follows.

Definition 1.8 The orbit of $\in \pi$ on $x \in X$ is $\pi^n(x)$: $n \in N$. If $(x, y) \in X$ are in the same orbit, then $\pi^n(x) = y \; for \; some \; n \in N$.

Example1.6 The orbits of σ in the above example are {1,2,3,4} and {5}.

Definition 1.9 A cycle is a permutation which contains at most one orbit with more than one element. The cycle $(i_1, i_2, ..., i_k)$ is such that the permutation π sends $i_1 \; to \; i_{j+1}$, for $1 \leq j \leq k-1$ and sends i_k back to i_1. (Nandakumar, 2010)

Definition 1.10 The length of a cycle is the number of elements in such cycle. We call a cycle, k-cycle, if there are k number of elements in that cycle and we say k is the length of the cycle. If k=1, then such permutation is the identity permutation (Adetunji, 2010)

For instance, if

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$$

in cycle form is written as (1234), then σ is called 4-cycle in S_4. Also

$$\rho = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 3 & 2 & 4 \end{pmatrix}$$

written as (254) is a 3- cycle in S_5.
Below are some remarks which give us more informations about the structure of group of permutations if they are written in cycle form.

Remark 1.6 The relation $x \sim y$ defined by $\sigma^n(x) = y, n \in N, \sigma \in S_n$ is an equivalence relation.
The implication of remark (1.6) is that any permutation group S_n is partitioned into disjoint classes of cycles.

Remark 1.7 i.) $x, y \in k - cycles$ if and only if $\sigma^n(x) = y$ where $1 \leq n \leq k$
ii.) Every permutation in S_n can be written as a product of disjoint cycles.

Definition 1.11 If $\sigma \in S_n$ is the product of disjoint cycles of length $q_1, q_2, q_3, ..., q_k$ such that $q_1 \leq q_2, \leq q_3 \leq ... \leq q_k$ then the integers $q_1, q_2, q_3, ..., q_k$ are called the cycle type of σ.

Definition 1.12 For any $\pi \in S_n$, the cycle type of π is an ordered list of the lengths of the cycle decomposition of π.

In example [1.5] , σ could be expressed as σ = (1423)(5) and its cycle type is (4,1). A cycle of length one in a permutation σ such as (4) of σ is called the fixed point and usually omitted from the cycle notation.

Remark 1.8 Let $\sigma \in S_n$ be a permutation, with cycle type $(q_1, q_2, q_3, ..., q_k)$ The order of σ is the least common multiple of $q_1, q_2, q_3, ..., q_k$.

Definition 1.13 A cycle of length two is called transposition.

Proposition 1.4 Every $\sigma \in S_n$ can be written as a product of transpositions (every permutation can be written as a product of adjacent transpositions, that is, transpositions of the form (i,i+1)).

Conjugation in a group

Definition 1.14 Let G be any group. If $g.x \in G$, we define the conjugates of g by x, by the element xgx^{-1}.
If $gh \in G$, and there are some $x \in G$ such that $xgx^{-1} = h$,we say g and h are conjugate in G.
For the group G, we define a relation ~ by $g \sim h$ if g and h are conjugate in G.
The set of all elements conjugate to a given g is called conjugacy class of g and we denote by $K_g = \{ h: h = xgx^{-1} \}$ for $x \in G$.

Proposition 1.5 Conjugacy is an equivalent relation. Thus, the distinct conjugacy classes partition G. (This partition is not in the sense of partition of integers).

Proposition 1.6 Let \$G\$ be any group, and let $xq_1, q_2, q_3, ..., q_n \in G$, then for any $n \geq 2$, the conjugate of $q_1, q_2, q_3, ..., q_n$ by x is the product of the conjugates by x of $q_1, q_2, q_3, ..., q_n$.

Proposition 1.7 Let G be an Abelian group, then for any $g \in G$, the conjugacy class of $g \in G$ is the singleton set {g}.

Conjugation in symmetric group S_n
We shall next briefly discuss conjugacy classes of the symmetric group S_n.

Example 1.7 In S_3, the conjugates of (12) is computed in the table below.

σ	(1)	(12)	(13)	(23)	(123)	(132)
$\sigma(123)\sigma^{-1}$	(12)	(12)	(23)	(13)	(23)	(13)

TABLE 2. Conjugacy class of (12) in S_3

The conjugates of (12) are in the second row: {(12),(13), (23)}
In a similar manner, the conjugacy class of (123) is { (123),(132)},
as Table 2 reveals.

σ	(1)	(12)	(13)	(23)	(123)	(132)
$\sigma(123)\sigma^{-1}$	(123)	(132)	(132)	(132)	(123)	(123)

TABLE 3. Conjugacy class of (123) in S_3

The effect of conjugation in S_3 which is true for $S_{n>0}$ is as follows:

Given $\pi = (123)$.What is $\sigma\pi\sigma^{-1}$?
Note that $\pi : 1 \rightarrow 2$

$$\sigma\pi\sigma^{-1}\big(\sigma(1)\big) = \sigma\pi(1) = \sigma(2)$$
$$\sigma\pi\sigma^{-1}\big(\sigma(2)\big) = \sigma\pi(2) = \sigma(3)$$
$$\sigma\pi\sigma^{-1}\big(\sigma(3)\big) = \sigma\pi(3) = \sigma(1)$$

So if, $\pi = (123)$ that is

$$1 \rightarrow 2 \rightarrow 3 \rightarrow 1$$

Then

$$\sigma\pi\sigma^{-1} = \sigma(1) \rightarrow \sigma(2) \rightarrow \sigma(3) \rightarrow \sigma(1)$$

So

$$\sigma\pi\sigma^{-1} = \sigma(1).\sigma(2).\sigma(3)$$

It is obvious from the above that conjugation preserves cycle structure.

Nilpotent orbits of type A

Let $M_n(C)$ be the set of all n x n matrices whose entries are in C. We denote the entries of any A $\in M_n(C)$ by a_{ij} and also write $A = [a_{ij}]$. We denote the identity and zero matrix in $M_n(C)$ by I_n and O_n respectively, and define E_{ij} with entry $e_{ij} = 1$ and zero elsewhere. E_{ij} form a basis of $M_n(C)$. Hence, is of complex dimension$dim_c M_n(C) = 2n^2$. $M_n(C)$ is a ring with the usual addition and multiplication of n x n matrices and I_n it's identity. $M_n(C)$ is not commutative for $n > 1$. The ring $M_n(C)$ acts on C^n from the left, giving C^n the structure of $M_n(C)$-module.

Definition 1.15 : Let A ∈ $M_n(C)$, A is said to be nilpotent if there exists integer K > 0 such that $A^k = 0$.

Definition 1.16. The nilpotent cone of the Lie algebra $gl_n(C)$ denoted by N, consists of all nilpotent elements in $gl_n(C)$.

Remark 1.9. nilpotency in light of definition {def1} could as well implies that the eigen values of such matrices are zero or their characteristic polynomials are the same as that of zero matrix .

Definition 1.17 Given $r > 0, \alpha \in R$, we denote by $J_r(\alpha)$ $an\, n \times n$ matrix of the form

$$J_r(\alpha) = \begin{pmatrix} \alpha & 1 & 0 & \cdots & 0 \\ 0 & \alpha & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \cdots & 1 \\ 0 & 0 & \cdots & 0 & \alpha \end{pmatrix} \tag{1.16}$$

where α are the eigenvalues and they appear at the main diagonal, 1 appears at the super-diagonal and zero elsewhere. $J_r(\alpha)$ is called Jordan block.
Since nilpotent matrices in $gl_n(C)$ have all eigenvalues equal to zero, in view of this, $J_r(\alpha)$ in (1.16) becomes

$$J_r(0) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \cdots & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \tag{1.17}$$

Proposition 1.7 (Kuttler, 2007)
Let $X \in gl_n(C)$ be nilpotent operator on C^n. Then there exists a basis for C^n such that the matrix X with respect to this basis is of the form

$$X_\lambda = \begin{pmatrix} J_{\lambda_1}(0) & & 0 & \\ & J_{\lambda_2}(0) & & \\ & & \ddots & \\ & 0 & & J_{\lambda_n}(0) \end{pmatrix} = \oplus J_{\lambda_i}(0)$$

Where $\lambda = (\lambda_1, \lambda_2, ..., \lambda_s)$ and $\sum_{i=1}^{s} \lambda_i = n$.

Definition 1.18. The orbits of the action of G on N for each X∈ N corresponding to $\lambda \in P(n)$ denoted by
$O_\lambda = \{g^{'} = gXg^{-1} : g \in gl_n(C)$.

The general linear group acts on N by conjugation. This leads to an equivalence relation in N which partitions N into disjoint classes, with each class being a nilpotent orbit determined by sizes of Jordan blocks a nilpotent matrix has. Listing the sizes of the blocks in descending order gives a certain partition of n. Thus the set of nilpotent orbits is in one to one correspondence with the set of partitions of integer n (P(n)). We denote by O_λ the nilpotent orbit that correspond to a partition λ and we say that each $X \in N$ is of Jordan type λ

2. Partition of integers and Cycle Structure of Elements of group of permutations

We recall that one of the interesting applications of group of permutations is revealed when they are expressed in cycle form. In actual sense, one of the connections between partition of integers and the group of permutations is revealed in the proposition below.

Proposition 2.1 The cycle type of elements of $S_n$are indexed by elements of P(n).

Proof. Now, let $\pi \in S_n$ with cycle decomposition of the form $(a_1, a_2, ..., a_{\lambda_2})(b_1, b_2, ..., b_{\lambda_1})...(g_1, g_2, ..., g_{\lambda_k})$, hence this permutation has cycle type $(\lambda_2, \lambda_1, ..., \lambda_k)$, since it is an ordered list, it could be written in any form. But one interesting thing to note here is that, the sum of all the lengths of the cycles , regardless of how it is written, must be equal to n. However, if we choose to write the lengths in a decreasing order, then we have a partition of n. Hence, the result.

Example 2.1. Let n=3, the table {2.1} reveals partitions of 3 and the associated cycle types of elements of S_3

$\pi \in S_3$	$\lambda \in P(3)$
(1)(2)(3)	(1, 1,1)
(1)(23)(12)(3)	(2, 1)
(321)(132)(123)	(1)

Cycles of elements of S_3.

Lemma 2.2 For any $n \in N$ there is one to one correspondence between P(n) and the conjugacy classes of S_n.

Proof :

We recall that the cycle type of elements of S_n are indexed by P(n) and each cycle structure of elements of S_n determined the conjugacy class of element of S_n. hence the result.

It is clear from Lemma (2.2) that, conjugacy classes of S_n are indexed by the elements of P(n). Now, suppose $n \in N$ and $\lambda \vdash n$ such that λ has m_i parts for each i in other words there are $m_1 1^{'s}, m_1\ 2^{'s}, m_3 3^{'s}$ and so on

Let K_λ be the conjugacy class in the symmetric group of degree n composing the elements whose cycle type is λ

Then

$$|K_\lambda| = \frac{n!}{\coprod (b_i)^{m_i}(m_i!)}.$$

Example 2.2. For n = 4, we have the following partitions:

$$\lambda = (4), \mu = (4,1), \upsilon = (2^2), \gamma = (2, 1^2), \varsigma = (1^2)$$

with

$$|K_\lambda| = \frac{4!}{4^1 \times 1!} = 6,\ |K_\mu| = \frac{4!}{3^1 \times 1!} = 8,\ |K_\upsilon| = \frac{4!}{2^2 \times 2!} = 3,\ |K_\Upsilon| = \frac{4!}{2 \times 1^2!} = 6,\ |K_\varsigma| = \frac{4!}{1^4 \times 4!} = 1,$$

In another direction, we consider the link between partition of integers and G module, where $G = S_n$ We recall that a G-module is said to be completely reducible if it is a direct sum of irreducible G-modules. For any finite group G, the number of inequivalent irreducible G-module is determined by the number of conjugacy classes of G. Now, if G = S_n, we know that its conjugacy classes consist of permutations of the same cycle type as determined by partitions of n. Therefore, the number of inequivalent irreducible S_n −module is the number of partitions of n.

For instance, the number, of irreducible S_4-module are just the cardinality of P(4)={(4), (3,1),(2,2), (2,1,1), (1,1,1,1)}. Since #P(4)=5, then there are 5 irreducible modules for S_4. In general, the irreducible S_n −module indexed by $\lambda \vDash n$ is usually denoted by S^{λ} and called the Specht module corresponding to λ.

3. Integer Partitions and Nilpotent Orbits

In this section we consider the classification of nilpotent orbits in $gl_n(C)$ under the action of $Gl_n(C)$. This we do in the framework of partition λ of integers n.
The general linear group acts on N by conjugation. This leads to an equivalence relation in N which partitions N into disjoint classes, with each class being a nilpotent orbit determined by sizes of Jordan blocks a nilpotent matrix has. Listing the sizes of the blocks in descending order gives a certain partition of n. Thus the set of nilpotent orbits is in one to one correspondence with the set of partitions of integer n (P(n)). We denote by O_{λ} the nilpotent orbit that correspond to a partition λ and we say that each $X \in N$ is of Jordan type λ.

Proposition 3.1 (Collingwood and McGovern, 1993). Let $X \in gl_n(C)$ be of Jordan type λ. There exists a bijection between the collection O_{λ} and the set P(n). The bijection is such that, each nilpotent element X is taken to the partition λdetermined by the block sizes in its Jordan canonical form.

Remark 3.1 From proposition (3.1), it is obvious that the zero orbits corresponds to the partition 1^n. In particular, the set of nilpotent orbits is finite .

Proposition 3.2 (Henderson, 2014) Any nilpotent matrix remain invariant under scalar multiplication, i.e if X is a nilpotent and $\kappa \neq 0$ then, $\kappa X$is nilpotent. X and κX are conjugate.
Remark 3.2 Following (3.2) , it is obvious that every nilpotent orbit in $gl_n(C)$ corresponds to a unique partition of n, that is, the number of nilpotent orbits in $gl_n(C)$ is at least |P(n)|, in other words, nilpotents orbits are separated by the uniqueness of the Jordan normal form. This gives the classification of nilpotent orbits in type A.N is referred to as nilpotent cone, since the property of nilpotency remain invariant under scalar multiplication.

Ordering on Nilpotent Orbits. Since there is a bijection between P(n) and the elements of Nilpotent cone ($X \in N$), hence, the ordering of $\lambda \in P(n)$ has a direct implication on the ordering of orbits in N. We recall that for $\mu, \lambda \in P(n)$, we say λ dominate μ if

$$\sum_{i=1}^{k} \lambda_i \geq \sum_{i=1}^{k} \mu_i \qquad 3.1$$

for some $k \in N$.

A direct conclusion from the ordering on P(n) to that of orbits in N is that, for $O_{\lambda}, O_{\mu} \subset N$, we say O_{μ} is contained in O_{λ} and write $O_{\mu} \subseteq O_{\lambda}$ if λ dominates μ.

The closure $\overline{O_{\lambda}}$ of the nilpotent orbit O_{λ} is the union of O_{λ} with other nilpotent orbits O_{μ}, such that $\lambda \trianglerighteq \mu$.

Theorem 3.3 [6]. Denote by O_{λ} the set of matrices conjugate to X_{λ} , then O_{μ} is contained in the closure $\overline{(O_{\lambda})}$ of O_{λ} if $\lambda \trianglerighteq \mu$. (i.e λ dominate μ).

Below are some types of nilpotent orbits as parameterized by partitions

i. There exists a unique Smallest orbit, that is, the 0-orbit which contains only 0. In this case, the corresponding partition $\lambda = 1^n$. Here λ is dominated by every other partitions of n.

ii. There exists also a unique maximal orbit called the regular orbit denoted by O_{rg} which is dense in N. Here λ = 1, and it is dominates every partition of n.

iii. There is a smallest orbit but larger than 0-orbit called the minimal orbit.

iv. We equally have a largest orbit which is smaller than the regular orbit, called the subregular orbit. In this case, $\lambda = n - 1,1$.

Example 3.1 Let $n = 5.0 - orbit = O_{1^5}$ and $O_{rg} = O_5$. Hence the inclusion below

$$O_{(1^5)} \subset O_{(2,1^3)} \subset O_{(2^2,1)} \subset O_{(3,1^2)} \subset O_{(3,2)} \subset O_{(4,1)} \subset O_{(5)} \quad . \tag{3.2}$$

Remark 3.3. The closure nilpotent orbits ($O_\lambda \subset N$) are nilpotent varieties. These are singular varieties for non-zero $X \in N$, i.e $\lambda \neq 1^n$. Also, it is note worthy that $\overline{O_{reg}} = N$'

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