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Common fixed point results for graph weak- quasi contraction mappings in b-metric spaces

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Abstract
In this article, we introduce a new class of graph weak- quasi contraction mappings. The existence of the unique common fixed point for a pair of an integral version of this map is hereby proved in the context of a b- metric space. These results are generalizations of related known results in metric spaces and b-metric spaces. Examples are given to validate the results obtained.

Keywords and Phrases: graph contraction, quasi contraction, common fixed point, b-metric space, graph weak-quasi contraction maps.

Introduction

Fixed point theory provides mathematical tools for solving functional equations in mathematics and applied mathematics. Banach (1992) introduced a contraction mapping that is applied to solve integral type problems. This result has been generalized and improved in various areas by many researchers, (Eke *et al*., (2019): Eke (2016): Eke *et al*., (2018): Olaleru (2009): Umudu *et al*.,(2020) for example. One of the most general of these contraction maps is called quasi contraction map introduced by Ciric (1971) in a metric space.

Definition 1.1 Ciric (1971): Let (X, d) be a metric space. Let f be a self map on X. Then f is called quasi contraction if it satisfies the following conditions:
$d(fx, fy) \leq q \max \{d(x, y), d(x, fx), d(y, fy), d(x, fy), d(y, fx)\}$,
for every $x, y \in X$ and $0 \leq q < 1$.

In 1974, Ciric (1974) introduced the completeness of quasi contraction operator in a metric space as follows;
Let f be a self mapping on a metric space (X, d). For $Y \subset X$ and for each $x \in X$we have
(i) $\delta(Y) = sup\{d(x, y): x, y \in Y\}$,
(ii) $O(x, n) = \{x, fx, f^2x, \cdots, f^nx\}$ for all $n \in N$,
(iii) $O(x, \infty) = \{x, fx, f^2x, \cdots\}$ for all $n \in N$.
A metric space (X, d) is called f- orbitally complete if every Cauchy sequence of $O(x, \infty)$ (for some $\in X$) converges to a point in X .

Now we give brief description of graph theory. Details of graph theory can also be found in Bondy and Murty (1976) for interested readers.
Let (X, d) be a metric space, $\delta = \delta(X)$ is the diagonal of X. Let V be a set and $E \subset V \times V$ be a binary relation on V, the ordered pair (V, E) is called a graph G. The elements of E are called edges and are denoted by $E(G)$ while the elements of V are called vertices and denoted by $V(G)$. If the edges are directed then we have a directed graph. Suppose the vertices cannot be connected by more than one edge, then G has no parallel edge. Thus, G can be represented by the pair $(V(G), E(G))$. If x and y are vertices of G, then a path of G from x to y of length $k \in N$ is a finite sequence $(x_n)_N, n \in (1,2,3,\cdots,k)$ of vertices such that $x = x_0, \cdots, x_k = y$ and $(x_n, x_{n-1}) \in E(G)$ for $n \in (1,2,3,\cdots,k)$. If there is a path between any two vertices then that graph is connected. The graph G is said to be weakly connected if $\overline{G}$ is connected where $\overline{G}$ denotes the undirected graph obtained from G by ignoring the direction of edges. Suppose G^{-1} is a graph obtained from G by reversing the direction of edges then we have

$$E(G^{-1}) = \{(x, y) \in X \times X: (y, x) \in E(G)\}.$$

$\overline{G}$ can be treated as a directed graph when the set of edges are symmetric to obtain
$E(\overline{G}) = E(G) \cup E(G^{-1})$.

In 2008, Jackymski (2008) introduced a generalization of Banach contraction known as G-contraction which is a contraction mapping with a graph structure defined on it.
Definition 1.2 Jackymski (2008): A mapping $f: X \to X$ is called G-contraction if f preserves edges of G i.e. for all $x, y \in X, (x, y) \in E(G)$ implies $(fx, fy) \in E(G)$ and f decreases weight of edges of G in the following way;
there exists $\alpha \in [0, 1)$ and for all $x, y \in X, (x, y) \in E(G)$ implies $d(fx, fy) \leq \alpha d(x, y)$.

In 1969, Nadler (1969) introduced a class of multivalued mapping which is defined on contraction mappings to obtain multivalued contraction mapping.
Definition 1.3 Nadler (1969): Let X and Y be nonempty sets. f is said to be multivalued mapping from X to Y if f is a function from X to the power set of Y. We denote the multivalued map by:

$$f: X \to 2^y.$$

Let (X, d) be a metric space and denote by $CB(X)$ the family of all nonempty, closed, and bounded subsets of X.
Definition 1.4 Nadler (1969): Let (X, d) be a metric space. We define the Hausdorff metric on $CB(X)$ induced by d. That is

$$H(A, B) = \max\{sup_{x \in A} d(x, B), sup_{x \in B} d(x, A)\},$$

for all $A, B \in CB(X)$, where $CB(X)$ denotes the family of all nonempty closed and bounded subsets of X and
$d(x, B) = inf\{d(x, b): b \in B\}$ for all $x \in X$.

Definition 1.5 Nadler (1969): Let (X, d) be a metric space. A map $f: X \to CB(X)$ is said to be multivalued contraction if there exists $0 \leq \alpha < 1$ such that $H(fx, fy) \leq \alpha d(x, y)$, for all $x, y \in X$.

In 1993, Czerwik (1993) generalized the concepts of a metric space by introducing a given real number to the triangle inequality axiom of the metric space and termed the space, b-metric space.

Definition 1.6 Czerwik (1993): Let X be a nonempty set and let $s \geq 2$ be a given real number. A function $d: X \times X \to$ is called a b-metric provided that for all $x, y, z \in X$,
(i) $d(x, y) = 0$ if and only if $x = y$,
(ii) $d(x, y) = d(y, x)$,
(iii) $d(x, z) \leq s\big(d(x, y) + d(y, z)\big)$.
A pair (X, d) is called a b-metric space.

Example 1.7 Czerwik (1993): Consider the set $X = [0,1]$ endowed with the function $d: X \times X \to$ defined by $d(x, y) = |x - y|^2$ for all $x, y \in X$. Here, $(X, d, 2)$ is a b-metric space but not a metric space.

Some researchers have proved fixed point and common fixed point theorems for multivalued mappings satisfying G-contractive conditions on a domain of sets endowed with a directed graph (see; Abbas (2015) for instance). Joseph *et al* (2016) and other researchers (Kubtbi *et al*., (2014): Qawaqueh *et al*., (2019)) proved several fixed and common fixed point results for multivalued contraction mappings in a complete b-metric spaces.

Fallahi and Rad (2020) proved the existence and uniqueness of fixed point for a Banach contractive type mapping in algebraic cone metric spaces associated with an algebraic distance and endowed with a graph. Acar *et al*. (2021) proved the existence and uniqueness of some fixed points for rational multivalued G- contraction and F- Khan -type multivalued contraction mappings on a metric space equipped with a graph.
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The establishment of fixed point theorems for single valued maps satisfying contractive conditions for integral type inequality was introduced by Branciari (2002). Liu *et al*. (2013) and Rhoades (2003) have more works in this regard. Benchabane *et al*. (2019) proved the existence of a unique common fixed point for multivalued integral type contraction mappings on a family of sets endowed with a graph.

Based on the works of Benchabane *et al*. (2019), we introduce in this paper, a new class of mapping called graph weak- quasi contraction mappings and prove a common fixed point theorems for multivalued integral version of this map in b-metric spaces endowed with a graph

2 Preliminaries

In this section, we recall some definitions and theorems that led to the development of our main results.
First we consider these two classes of functions.

Definition 2.1 Benchabane *et al.* (2019): The class φ consists of nondecreasing continuous functions $\varphi: [0, +\infty) \to [0, +\infty)$ such that $\varphi(0) = 0$ and φ is sub-additive, i.e. for every $t_1, t_2 \in R^{+,}$
$\varphi(t_1 + t_2) \leq \varphi(t_1) + \varphi(t_2)$.

Definition 2.2 Benchabane *et al.* (2019): The class μ is the set of functions $\vartheta: [0, +\infty) \to [0, +\infty)$ which satisfy the following conditions:
(i) ϑ is Lebesgue integrable and summable on each compact subset $[0, +\infty)$,
(ii)$\int_0^{\varepsilon} \vartheta(t)\, dt > 0$ for each $\varepsilon > 0$.

We hereby recall some lemmas that will be useful in the sequel.

Lemma 2.3 Acar *et al.* (2021): Let $(r_n)_n$be a nonnegative sequence and $\vartheta \epsilon \mu$. Then
$\lim_{n\to\infty} \int_0^{r_n} \vartheta(t)\, dt = 0$ if and only if $\lim_{n\to\infty} r_n = 0$.

Lemma 2.4 Stojakvoic *et al.* (2015): For every $\vartheta \epsilon \mu$, we have
$\int_0^{a+b} \vartheta(t)\, dt \leq \int_0^{a} \vartheta(t)\, dt + \int_0^{b} \vartheta(t)\, dt,$ for all $a, b \geq 0$.

Lemma 2.5 Nadler (1969): If $A, B \in CB(X)$ with $H(A, B) < \varepsilon$, then for each $a \in A$, there exists an element $b \in B$ such that $d(a, b) < \varepsilon$.

Definition 2.6 Czerwik (1993): Let A and B be two nonempty subsets of . Then
(a) there is an edge between A and B, means there is an edge between some $a \in A$ and $b \in B$ which we denote by $(A, B) \subset E(G)$.
(b) there is a path between A and B, means there is a path between some $a \in A$ and $b \in B$ which we denote by $(A, B) \subset E(G)$.

In$CB(X)$, we define a relation R in the following way:
for $A, B \in CB(X)$, ARB if and only if there is a path between A and B .
We say that a relation R on $CB(X)$ is transitive whenever there is a path between A and B and there is a path between B and C, there is a path between A and C .

Definition 2.7 Benchabane *et al.* (2019): Let $f: CB(X) \to CB(X)$ be a multivalued mapping. The set $A \in CB(X)$ is said to be a fixed point of f if $f(A) = A$. The set of all fixed points of f is denoted by $F(f)$.
Consider $X_f = \{U \subset CB(X): (U, f(U)) \subset E(G)\}$.
A subset A of $CB(X)$ is said to be complete if for any set $X, Y \in A$, there is an edge between X and Y.
We state some theorems that motivate our research.
Abbas *et al.* (2015) proved fixed point theorem for multivalued self mapping on $CB(X)$ satisfying certain graph μ-contractive conditions in a complete metric space endowed with a

directed graph. The authors in Abbas *et al*. (2015) utilized the following property in proving their result.

Definition 2.8 : A graph G is said to have property (P*) if for any sequence $\{X_n\}_n$ in $CB(X)$ with $X_n \to X$, as $n \to \infty$, the existence of an edge between X_n and X_{n+1} for $n \in N$ implies the existence of a subsequence $(X_{n_k})_k$ of X_n with an edge between X_{n_k} and X, for $k \in N$.

Benchabane *et al*. (2019) generalized the result of Abbas *et al*. (2015) by introducing the following graph (φ, μ)-weak contraction maps and proved some fixed point results.

Definition 2.9 Benchabane *et al.* (2019): Let (X, d) be a metric space endowed with a directed graph G such that $V(G) = X$ and $\delta \subset E(G)$. Let $S, T: CB(X) \to CB(X)$ be two multivalued mappings. The pair (S, T) of maps is said to be graph φ, μ-weak contraction maps if ;
(i) for every U in $CB(X)$, $(U, SU) \subset E(G)$ and $(U, TU) \subset E(G)$,
(ii) there exists a nondecreasing function $\mu: R^+ \to R^+$with $\sum_{n=0}^{\infty} \mu^i(t)$ is convergent for all $t > 0$, $\vartheta \epsilon \mu$, $\varphi \in \rho$ and $L \geq 0$ such that if there is an edge between A and B with $S(A) \neq T(B)$, then

$$\varphi\left(\int_0^{H(S(A),T(B))} \vartheta(t)\, dt\right) \leq \mu\left(\varphi\left(\int_0^{M_{(S,T)(A,B)}} \varphi(t) dt\right)\right) + L\int_0^{N_{S,T(A,B)}} \varphi(t) dt\,,$$

where

$$M_{S,T}(A, B) = max\{H(A, B), H(A, S(A)), H(B, T(B)), \frac{H(A, T(B)) + H(B, S(A))}{2}\}$$

and

$$N_{S,T}(A, B) = min\{H(A, S(A)), H(B, T(B)), H(A, T(B)), H(B, S(A))\}.$$

The following is the established result.

Theorem 2.10 Benchabane *et al.* (2019): Let (X, d) be a metric space endowed with a directed graph G such that $V(G) = X$ and $\delta \subset E(G)$, the relation R on $CB(X)$ is transitive and $S, T: CB(X) \to CB(X)$ is a graph φ, μ -weak contraction maps. Then the following statements hold.
(i) $F(S)$ or $F(T) \neq \emptyset$ if and only if $F(S) \cap F(T) \neq \emptyset$,
(ii)$F(S) \cap F(T) \neq \emptyset$ provided that G is weakly connected and satisfies the property (P*),
(iii) If $F(S) \cap F(T)$ is complete, then the Pompeiu- Hausdorff weight assigned to $U, V \in F(S) \cap F(T) \neq \emptyset$,
(iv) $F(S) \cap F(T)$ is complete if and only if $F(S) \cap F(T)$ is a singleton.

3. **Fixed point results in a b - metric space**

In this section, we introduce new class of mappings called graph weak- quasi contraction maps in a b- metric space endowed with directed graph. The common fixed point for two multivalued self mappings satisfying integral type version of this new map in a b-metric space endowed with directed graph is proved. The consequences of our theorem are provided with example to support our results.

First, we give the definition of our map.

Definition 3.1: Let (X, d) be a metric space endowed with a directed graph G such that $V(G) = X$ and $\delta \subset E(G)$. Let $S, T: CB(X) \rightarrow CB(X)$ be two multivalued mappings. The pair (S, T) of maps is said to be graph weak- quasi contraction maps if ;

(i) for every U in $CB(X)$, $(U, SU) \subset E(G)$ and $(U, TU) \subset E(G)$,

(ii) there exists a nondecreasing function $\mu: R^{+\rightarrow R^+}$ with $\sum_{n=0}^{\infty} \mu^i(t)$ with is convergent for all $t > 0$, $\vartheta \epsilon \mu$, $\varphi \in \rho$ and$L \geq 0$ such that if there is an edge between A and B with $S(A) \neq T(B)$, then

$$\varphi\left(\int_0^{H(S(A),T(B))} \vartheta(t)\, dt\right) \leq \mu\left(\varphi\left(\int_0^{M_{S,T(A,B)}} \varphi(t) dt\right)\right) + L\int_0^{N_{S,T(A,B)}} \varphi(t) dt\,,$$

where

$$M_{S,T}(A, B) = max\{H(A, B), H(A, S(A)), H(B, T(B)), H(A, T(B)), H(B, S(A))\}$$

and

$$N_{S,T}(A, B) = min\{H(A, S(A)), H(B, T(B)), H(A, T(B)), H(B, S(A))\}$$

Remark 3.2 :

(1) From condition (ii) in definition 3.1, if $max\{H(A, B), H(A, S(A)), H(B, T(B)), H(A, T(B)), H(B, S(A))\} = H(A, B)$ and $\varphi\left(\int_0^{H(S(A),T(B))} \vartheta(t)\, dt\right) = H(S(A), T(B))$ then we obtain the results of Abbas *et al*.(2015)

(2) If we replace the quasi contraction with $\{H(A, B), H(A, S(A)), H(B, T(B)), \frac{H(A,T(B))+H(B,S(A))}{2}\}$, then we have the result of Benchabane *et al*.(2019).

Remark 3.3 : (1) In a b-metric space, if the real number $s = 1$ then we obtain metric space. Since $s \geq 2$, then b-metric space generalizes metric spaces and the results obtained in b-metric spaces generalized the results in metric spaces.

(2) Definition 3.1 is also valid for b-metric space since every metric space is a b-metric space but the converse is not.

Now we state and prove that the existence of common fixed point for a pair of multivalued integral type satisfying graph weak- quasi contraction condition in b-metric space endowed with a directed graph is a singleton.

Theorem 3.4 : Let (X,d) be a b- metric space with a real number $s \geq 2$ endowed with a directed graph G such that $V(G)=X$ and $\delta \subset E(G)$,the relation R on $CB(X)$ is transitive and $S,T: CB(X) \rightarrow CB(X)$ is a graph weak - quasi contraction pair. Then the following conditions hold.
(i) $F(S)$ or $F(T) \neq \emptyset$ if and only if $F(S) \cap F(T) \neq \emptyset$,
(ii)$F(S) \cap F(T) \neq \emptyset$ provided that G is weakly connected and satisfies the property (P*),
(iii) If $F(S) \cap F(T)$ is orbitally- complete, then the Pompeiu- Hausdorff weight assigned to $U,V \in F(S) \cap F(T) \neq \emptyset$,
(iv) $F(S) \cap F(T)$ is orbitally complete if and only if $F(S) \cap F(T)$ is a singleton.
Proof: Suppose that $F(S) \neq \emptyset$. By assumption $(U,SU) \subset E(G)$. To prove that $U \in F(T)$ we need to assume that $U \neq F(T)$. Since the pair (S,T) is graph weak - quasi contraction and $(U,U) \subset E(G)$ then

$$\varphi\left(\int_0^{H(U,TU)} \vartheta(t)\,dt\right) = \varphi\left(\int_0^{H(SU,TU)} \vartheta(t)\,dt\right) \leq \mu\left(\varphi\left(\int_0^{M_{S,T(U,U)}} \vartheta dt\right)\right) + L\int_0^{N_{S,T(U,U)}} \vartheta(t)dt$$

$$\leq \mu\left(\varphi\left(\int_0^{M_{S,T(U,U)}} \vartheta(t)dt\right)\right) + L\int_0^{N_{S,T(U,SU)}} \vartheta(t)dt \quad \mu\left(\varphi\left(\int_0^{M_{S,T(U,U)}} \varphi(t)dt\right)\right)$$

Where

$$M_{S,T}(U,U) = max\{H(U,U), H\big(U,S(U)\big), H\big(U,T(U)\big), H\big(U,T(U)\big), H\big(U,S(U)\big)\}$$
$$= max\{H(U,U), H(U,U), H\big(U,T(U)\big), H\big(U,T(U)\big), H(U,U)\}$$
$$= H\big(U,T(U)\big)$$

By property of μ, we obtain

$$\varphi\left(\int_0^{H(U,TU)} \vartheta(t)\,dt\right) \leq \mu\left(\varphi\left(\int_0^{H(U,T(U))} \vartheta(t)dt\right)\right) < \varphi\left(\int_0^{H(U,T(U))} \vartheta(t)dt\right)$$

A contradiction. Thus $U \in F(T)$ which implies that$(U,TU) \subset E(G)$.
(ii) Let $A_0 \in CB(X)$ be arbitrary choosen. Assume $A_0 \in F(S)$ or $A_0 \in F(T)$ then $F(S) \cap F(T) \neq \emptyset$,. Suppose $A_0 \neq F(S)$ and $A_0 \neq F(T)$ then by definition of graph weak- quasi contraction we have $\big(A_0, S(A_0)\big) \subset E(G)$, this implies there exists some $x_0 \in A_0$ such that there is an edge between x_0 and some $x_1 \in S(A_0)$. Let $A_1 \in S(A_0)$ then by definition, implies that there is an edge between x_1 and some $x_2 \in T(A_1)$. Thus $A_2 \in T(A_1)$. By induction, we construct a sequence $(A_n)_n$such that $A_{2n+1} = S(A_{2n})$, $A_{2n+2} = T(A_{2n+1})$ and $(A_n, A_{n+1}) \subset \vartheta(t)dtE(G)$ for all $n \epsilon N$.
Observe that we assumed$A_{2n} \neq A_{2n+1}$, otherwise $A_{2n} = A_{2n+1}$, for some $n \in N$.
$S(A_{2n}) = A_{2n} = A_{2n+1}$, and therefore $A_{2n} \in F(S)$. By (i) $A_{2n} \in F(S) \cap F(T)$. Since the pair (S,T) is a graph weak- quasi contraction and $(A_{2n}, A_{2n+1}) \subset E(G)$ we have

$$\varphi\left(\int_0^{H(A_{2n+1},A_{2n+2})} \vartheta(t)\,dt\right) = \varphi\left(\int_0^{H(SA_{2n},TA_{2n+1})} \vartheta(t)\,dt\right)$$
$$\leq \mu\left(\varphi\left(\int_0^{M_{S,T(A_{2n},A_{2n+1})}} \vartheta dt\right)\right) + L\int_0^{N_{S,T(A_{2n},A_{2n+1})}} \vartheta(t)dt$$

$$\leq \mu\left(\varphi\left(\int_0^{M_{S,T(A_{2n},A_{2n+1})}} \vartheta dt\right)\right) + L\int_0^{N_{S,T(S(A_{2n}),\ A_{2n+1})}} \vartheta(t)dt \qquad \mu = \left(\varphi\left(\int_0^{M_{S,T(A_{2n},A_{2n+1})}} \vartheta dt\right)\right)$$

Where

$$M_{S,T(A_{2n},A_{2n+1})} = max\{H\max\{\mathrm{H}(A_{2n},\ A_{2n+1}), H(A_{2n},\ S(A_{2n})), H(A_{2n+1},\ T(A_{2n+1})),$$

$$H(A_{2n},\ T(A_{2n+1})), H(A_{2n+1},\ S(A_{2n}))\}$$
$$= max\{H \max\{\mathrm{H}(A_{2n},\ A_{2n+1}), H(A_{2n+1},\ A_{2n+2}), H(A_{2n},\ A_{2n+2})\}$$

$$= \delta(O_{S,T}(A,n)), \text{ for each } \in N\ .$$

where

$$\delta(O_{S,T}(A,n)) = \max\{\mathrm{H}(A_{2n},\ A_{2n+1}) : n \in N\}$$

Note that the functions belonging to S, T are smaller unit so we consider $M_{S,T}(A|2n, A_{2n+1}) = H(A|2n, A_{2n+1})$.
A situation where $M_{S,T}(A|2n, A_{2n+1}) = HA_{2n},\ A_{2n+1})$ is impossible. For

$$\varphi\left(\int_0^{H(A_{2n+1},A_{2n+2})} \vartheta(t)\, dt\right) \leq \mu\left(\varphi\left(\int_0^{M_{S,T}(A_{2n},A_{2n+1})} \vartheta dt\right)\right) = \mu\left(\varphi\left(\int_0^{H(A_{2n+1},\ A_{2n+2})} \vartheta dt\right)\right)$$

By property of μ we have

$$\varphi\left(\int_0^{H(A_{2n+1},\ A_{2n+2})} \vartheta dt\right) \leq \varphi\left(\int_0^{H(A_{2n+1},\ A_{2n+2})} \vartheta dt\right)$$

a contradiction. Thus we conclude that

$$\varphi\left(\int_0^{H(A_{2n+1},\ A_{2n+2})} \vartheta dt\right) \leq \mu\left(\varphi\left(\int_0^{H(A_{2n},\ A_{2n+1})} \vartheta dt\right)\right)$$

Similarly,

$$\varphi\left(\int_0^{H(A_{2n+2},\ A_{2n+3})} \vartheta dt\right) \leq \mu\left(\varphi\left(\int_0^{H(A_{2n+1},\ A_{2n+2})} \vartheta dt\right)\right)$$

For all $n \in N$ we have

$$\varphi\left(\int_0^{H(A_{n+1},\ A_{n+2})} \vartheta dt\right) \leq \mu\left(\varphi\left(\int_0^{H(A_n,\ A_{n+1})} \vartheta dt\right)\right)$$

Continuing the process we have

$$\varphi\left(\int_0^{H(A_{n+1},\ A_{n+2})} \vartheta dt\right) \leq \mu\left(\varphi\left(\int_0^{H(A_n,\ A_{n+1})} \vartheta dt\right)\right) \leq \vdots$$
$$\leq \mu^n\left(\varphi\left(\int_0^{H(A_0,\ A_1)} \vartheta dt\right)\right).$$

This shows that the sequence $(A_n)_n$ converges.
Next we prove that the sequence is Cauchy in CB(X). For $m > n$ and with the triangle inequality we have

$$\varphi\left(\int_0^{H(A_n,\ A_m)}\vartheta dt\right) \le \mu\left(\varphi\left(\int_0^{(s(H(A_n,\ A_{n+1})+H(A_{n+1},\ A_m))}\vartheta dt\right)\right)$$

$$\le \mu\left(\varphi\left(\int_0^{sH(A_n,\ A_{n+1})+s(sH(A_{n+1},\ A_{n+2})+H(A_{n+2},\ A_m))}\vartheta dt\right)\right)$$

$$\le \mu\left(\varphi\left(\int_0^{sH(A_n,\ A_{n+1})+s^2H(A_{n+1},\ A_{n+2})+s^3H(A_{n+2},\ A_{n+3})+\cdots+s^{m-1}H(A_{m-1},\ A_m))}\vartheta dt\right)\right)$$

$$\le \mu\left(\varphi\left(\sum_{i=n}^{m-1}\sum_{j=n}^{m-1}\int_0^{s^jH(A_i,\ A_{i+1}))}\vartheta dt\right)\right) \le \mu\left(\sum_{i=n}^{m-1}\sum_{j=n}^{m-1}(\varphi\int_0^{s^jH(A_i,\ A_{i+1}))}\vartheta dt)\right)$$

$$\le \sum_{i=n}^{m-1}\sum_{j=n}^{m-1}\mu(\varphi(\int_0^{s^jH(A_i,\ A_{i+1}))}\vartheta dt))$$

By taking the limit as $n, m \to \infty$ we obtain $\varphi\left(\int_0^{s^jH(A_i,\ A_{i+1})}\vartheta\, dt\right) \to 0$, then $\int_0^{H(A_n,\ A_m)}\vartheta dt \to 0$ as $n, m \to \infty$. By Lemma 2.3, $H(A_n,\ A_m) \to 0$ as $n, m \to \infty$. Therefore $(A|n)_n$ is a Cauchy sequence in $CB(X)$. Since (X, d) is complete then it implies $CB(X)$ is complete. Thus $(A_n)_n$ converges to a point $V \in CB(X)$ as $n \to \infty$.

To prove the existence of the common fixed point of S and T, that is, $V = SV = TV$. We assume $V = SV$ and prove $V = T$ V. On the contrary, suppose $V \neq T$ V, since $(A|2n, A_{2n+1}) = (A|2n, T(A_{2n}))\complement E(G)$ for all $n \in N$. By property (P*), there exists a subsequence $(A|2n_k)_k$ of $(A|2n)_n$such that there is an edge between $(A|2n_k)$ and V for every $k \in N$. Since the pair (S, T) is a graph weak- quasi contraction and$(V, A_{2n+k})\complement E(G)$, we have

$$\varphi\left(\int_0^{H(A_{2n_{k+1}},\ T(V))}\vartheta dt\right) = \varphi\left(\int_0^{H(S(A_{2n_k},\ T(V))}\vartheta dt\right)$$

$$\le \mu\left(\varphi\left(\int_0^{M_{S,T}\left(A_{2n_k},\ V\right)}\vartheta dt\right)\right) + L\int_0^{N_{S,T}\left(A_{2n_k},\ V\right)}\vartheta dt$$

$$\le \mu\left(\varphi\left(\int_0^{M_{S,T}\left(A_{2n_k},\ V\right)}\vartheta dt\right)\right) + L\int_0^{H(S\left(A_{2n_k}\right),\ V)}\vartheta dt$$

$$\le \mu\left(\varphi\left(\int_0^{M_{S,T}\left(A_{2n_k},\ V\right)}\vartheta dt\right)\right) + L\int_0^{H\left(A_{2n_{k+1}},\ V\right)}\vartheta dt$$

where,

$$M_{S,T}(A_{2n_k}, \ V) = \max\{ H(A_{2n_k}, \ V), H\left(A_{2n_k}, S(A_{2n_k})\right), H(V,TV), H(A_{2n_k}, T\,V), H(S(A_{2n_k}), V)$$
$$= \max\{ H(A_{2n_k}, \ V), H(A_{2n_k}, A_{2n_{k+1}}), H(V,TV), H(A_{2n_k}, T\,V), H(A_{2n_{k+1}}, V)$$

Since there is an edge between A_{2n_k} and V and taking the limit as $k \to \infty$ we have

$$M_{S,T}(V,V) = H(V,TV).$$

Now applying the property of μ and φ we obtain

$$\varphi\left(\int_0^{H(V,TV)} \vartheta dt\right) \leq \mu\left(\varphi\left(\int_0^{H(V,TV)} \vartheta dt\right)\right) < \varphi\left(\int_0^{H(V,TV)} \vartheta dt\right)$$

This is a contradiction. Thus $V = TV$ and by (i) $F(S) \cap F(T) \neq \emptyset$.

(iii) Suppose that $F(S) \cap F(T)$ is orbitally complete. Let $U, V \in F(S) \cap F(T)$ and assume that $H(U,V) \neq \emptyset$. Since the pair (S,T) is a graph weak- quasi contraction, we obtain

$$\varphi\left(\int_0^{H(U,V)} \vartheta dt\right) = \varphi\left(\int_0^{H(SU,TV)} \vartheta dt\right) \leq \mu\left(\varphi\left(\int_0^{M_{S,T(U,V)}} \vartheta dt\right)\right) + L\int_0^{N_{S,T(U,V)}} \vartheta dt$$
$$\leq \mu\left(\varphi\left(\int_0^{M_{S,T(U,V)}} \vartheta dt\right)\right) + L\int_0^{H(SU,U)} \vartheta dt = \mu\left(\varphi\left(\int_0^{M_{S,T(U,V)}} \vartheta dt\right)\right)$$

Where,

$$M_{S,T}(U,V) = max\,\{H(U,V), H\big(U,S(U)\big), H(V,TV), H(U,TV), H(U|,V)$$
$$= max\,\{H(U,V), H(U,U), H(V,V), H(U,V), H(U,V) = H(U,V)$$

By the property of μ and φ, we have

$$\varphi\left(\int_0^{H(U,V)} \vartheta dt\right) \leq \mu\left(\varphi\left(\int_0^{H(U,V)} \vartheta dt\right)\right) < \varphi\left(\int_0^{H(U,V)} \vartheta dt\right)$$

a contradiction.

(iv) Suppose $F(S) \cap F(T)$ is orbitally complete. Let $U, V \in CB(X)$ be such that $U, V \in F(S) \cap F(T)$. By (iii) we have $H(U,V) = 0$. This shows that $F(S) \cap F(T)$ is a singleton. Conversely, suppose that $F(S) \cap F(T)$ is singleton. Since $\delta \complement E(G)$, then $F(S) \cap F(T)$ is orbitally complete. This ends the prove.

We hereby give an example to support our result.

Example 3.5: Let $X = \{1,2,3,\cdots,n\}$ (with $n > 4$) be the set of integers endowed with the b-metric $d: X^2 \to$ for real number $s = 2$ defined by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ \frac{1}{n+2} & \text{if } \quad \mathrm{x,y} \in [1,2,3,,4,5), x \neq y \\ \frac{n+3}{n+4} & \text{if otherwise} \end{cases}$$

The Pompeiu- Hausdorff metric is defined by

$$H(A,B) = \begin{cases} 0 \text{ if } A = B \\ \dfrac{1}{n+2} & \text{if } \quad \mathrm{A,B} \in [1,2,3,,4,5), A \neq B \\ \dfrac{n+3}{n+4} & \text{if otherwise} \end{cases}$$

Define the graph $G = (V(G), E(G))$ with $V(G) = X$ and $E(G) = \{i, j \in X \times X : i \leq j\}$. Let $S, T: CB(X) \to CB(X)$ be define by,

$$S(U) = \begin{cases} \{1,2,3\} \text{ if } U \in \{1,2,3,4,5\} \\ \{4,5\} \text{ if } \mathrm{U} \in \{6,7\} \\ \{1,2,3,4,5\} \text{ if otherwise} \end{cases}$$

$$T(U) = \begin{cases} \{1,2,3\} \quad \text{if} \quad \mathrm{U} \in \{1,2,3,4,5\} \\ \{4\} \text{ if U is not in} \{1,2,3,4,5\} \end{cases}$$

Define the mappings $\mu, \varphi, \vartheta: [0, \infty) \to [0, \infty)$ by $\mu(t) = \frac{3t}{2t+3}$, $\vartheta(t) = t$ and

$$\mu = \begin{cases} t^2 \text{ if } \mathrm{t} \in [0, \frac{1}{2}) \\ \frac{1}{2} \quad \text{if } \mathrm{t} \in \left(\frac{1}{2}, \infty\right) \end{cases}$$

Given $L > 0$, then S and T form a graph weak- quasi contraction and $\{1,2,3,\}$ is the unique commmon fixed point of S and T.

We deduce the following results from our main theorem.

Corollary 3.6 : Let (X, d) be a b- metric space with a real number $s \geq 2$ endowed with a directed graph G such that $V(G) = X$ and $\delta \complement E(G)$. Suppose the mapping $S, T: CB(X) \to CB(X)$ satisfy the following conditions.

(i) for every U in $CB(X)$, $(U, SU) \complement E(G)$ and $(U, TU) \complement E(G)$,

(ii) there exists a nondecreasing function $\mu: R^{+ \to R^{+}}$ with $\sum_{n=0}^{\infty} \mu^{i}(t)$ is convergent for all $t > 0$, $\vartheta \epsilon \mu$, $\mu \epsilon \rho$ and $L \geq 0$ such that if there is an edge between A and B with $S(A) \neq T(B)$, then

$$\varphi(H(S(A), T(B)) \leq \mu(\varphi(M_{S,T}(A,B))) + LN_{S,T}(A,B).$$

If the relation R on $CB(X)$ is transitive, then the following conditions hold.

(i) $F(S)$ or $F(T) \neq \emptyset$ if and only if $F(S) \cap F(T) \neq \emptyset$,

(ii)$F(S) \cap F(T) \neq \emptyset$ provided that G is weakly connected and satisfies the property (P*),
(iii) If $F(S) \cap F(T)$ is orbitally- complete, then the Pompeiu- Hausdorff weight assigned to $U, V \in F(S) \cap F(T) \neq \emptyset$,
(iv) $F(S) \cap F(T)$ is orbitally complete if and only if $F(S) \cap F(T)$ is a singleton.

Corollary 3.7 : Let (X, d) be a b- metric space with a real number $s \geq 2$ endowed with a directed graph G such that $V(G) = X$ and $\delta \complement E(G)$. Suppose the mapping $S: CB(X) \to CB(X)$ satisfy the following conditions.
(i) for every U in $CB(X)$, $(U, SU) \complement E(G)$,
(ii) there exists a nondecreasing function $\mu: R^{+ \to R^{+}}$ with $\sum_{n=0}^{\infty} \mu^{i}(t)$ is convergent for all $t > 0$, $\vartheta \epsilon \mu$, $\mu \epsilon \rho$ and $L \geq 0$ such that if there is an edge between A and B with $S(A) \neq S(B)$, then

$$\varphi\left(\int_{0}^{H(S(A),S(B))} \vartheta(t)\, dt\right) \leq \mu\left(\varphi\left(\int_{0}^{M(A,B)} \varphi(t) dt\right)\right) + L \int_{0}^{N(A,B)} \varphi(t) dt\,,$$

where

$$M(A,B) = max\{H(A,B), H(A,S(A)), H(B,S(B)), H(A,S(B)), H(B,S(A))\}$$

and

$$N(A,B) = min\{H(A,S(A)), H(B,S(B)), H(A,S(B)), H(B,S(A))\}$$

If the relation R on $CB(X)$ is transitive, then the following conditions hold.
(i)$F(S) \neq \emptyset$provided that G is weakly connected and satisfies the property (P*),
(ii) If $F(S)$ is orbitally- complete, then the Pompeiu- Hausdorff weight assigned to $U, V \in F(S)$ is 0,
(iii) $F(S)$ is orbitally complete if and only if $F(S)$ is a singleton.

Remarks 3.8: (i) Corollary 3.6 generalizes the result of Abbas *et al.*(2015) if the graph weak-quasi contraction map is replaced with contraction map and the real number $s = 1$ in b-metric space.
(ii) Theorem 3.4 generalized the result of Benchabane *et al.* (2019) in the sense that a more general contraction maps is used to replace the contractive map of the results in Benchabane *et al.* (2019) and our theorem is proved in b-metric space setting which equally generalizes the metric space.

Conclusion

In this work, a new class of mapping known as graph weak- quasi contraction mappings is introduced. Furthermore, existence of a unique common fixed point of a pair of multivalued integral type of this map is proved in b- metric spaces. Example is equally provided to support our results. This new class of mapping can be prove in different abstract spaces thereby widening the scope of study for contractive maps in fixed point theory by interested researchers.

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